

C^* -ALGEBRAS OF LABELLED GRAPHS III - K -THEORY COMPUTATIONS

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ABSTRACT. In this paper we give a formula for the K -theory of a labelled graph algebra when the labelled graph in question is left-resolving. We also establish strong connections between the various classes of C^* -algebras which are associated with shift spaces and labelled graph algebras. Hence by computing the K -theory of a labelled graph algebra we are providing a common framework for computing the K -theory of graph algebras, ultragraph algebras, Exel-Laca algebras, Matsumoto algebras and the C^* -algebras of Carlsen. We provide an inductive limit approach for computing the K -groups of an important class of labelled graph algebras, and give examples.

1. INTRODUCTION

The purpose of this paper is to provide formula for and a practical method of computing the K -theory of a labelled graph C^* -algebra for a large class of labelled graphs which, in particular, contains all left-resolving labelled graphs. To do this we first realise these labelled graph C^* -algebras as Cuntz-Pimsner algebras and then use the results of [19] to identify the K -groups in question as the kernel and cokernel of a certain map. Then, for an important class of labelled graph C^* -algebras, we give a procedure for computing these groups through an inductive limit process. The motivation for our computations is to develop the relationship between certain dynamical invariants of shift spaces and K -theoretical invariants of C^* -algebras associated to these shift spaces. This connection was first brought to light in the work of Cuntz and Krieger.

In [11] Cuntz and Krieger showed how to associate a C^* -algebra \mathcal{O}_A to a finite 0-1 matrix A with no zero rows or columns provided that the matrix satisfied a certain condition called (I). In [12, Proposition 3.1] it was shown that the K -groups of a Cuntz-Krieger algebra are isomorphic to the Bowen-Franks groups of the shift of finite type X_A associated to A (see [6]). Thus, a deep connection was established between the combinatorially-defined C^* -algebra \mathcal{O}_A and the (one-sided) shift of finite type X_A . Several generalisations of Cuntz-Krieger algebras have now been widely studied.

Combining the universal algebra approach of [1] and the graphical approach to Cuntz-Krieger algebras begun in [14], graph algebras were introduced in [24]. Graph algebras were originally defined for graphs satisfying a finiteness condition – the need for this condition was removed by Fowler and Raeburn in [17] (see also [16]). Using a different approach, Exel and Laca showed how to associate a C^* -algebra to an infinite 0-1 matrix with no zero rows or columns in [15]. A link between graph algebras and Exel-Laca algebras was provided by the ultragraph algebras introduced by Tomforde (see [37, 38]).

Motivated by the symbolic dynamical data contained in a Cuntz-Krieger algebra, Matsumoto provided a generalisation of Cuntz-Krieger algebras by associating to an arbitrary two-sided shift space Λ over a finite alphabet a C^* -algebra \mathcal{O}_Λ (see [26, 27, 28, 29, 31, 32]). Later Carlsen and Matsumoto modified this construction (see [8, 34]), and Carlsen further modified the definition of \mathcal{O}_Λ and extended it to one-sided shift spaces in [7] (see [9] for a discussion of the

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relationship between the three different definitions of \mathcal{O}_Λ and for an alternative construction of the C^* -algebra constructed in [7]). The results of [27, 28, 30, 33, 34] give connections between certain dynamical invariants of Λ and K -theoretical invariants of \mathcal{O}_Λ in the same spirit as [12].

By adapting the left-Krieger cover construction given in [22], any shift space over a countable alphabet may be presented by a left-resolving labelled graph. Hence, the labelled graph algebras introduced in [4] provide a method for associating a C^* -algebra to a shift space over a countable alphabet. As we shall see in Section 7, the class of labelled graph algebras contains the class of graph algebras and all of the classes of C^* -algebras discussed above. By computing the K -theory of a labelled graph algebra we will therefore be providing a unified approach to computing the K -theory of this wide collection of C^* -algebras.

The paper is organised as follows: Section 2 contains some background on labelled graphs, labelled spaces and their C^* -algebras. In Section 3 we introduce some notation and results which will be used in the rest of the paper. In Section 4 we introduce and discuss the class of *regular* weakly left-resolving labelled spaces. In Section 5 we give our key result, Theorem 5.6, which shows that the C^* -algebra of a regular weakly left-resolving labelled space may be realised as a Cuntz-Pimsner algebra in the sense of [19]. In Section 6 we use the general results of [19] to provide a formula for the K -theory of the C^* -algebra of a regular weakly left-resolving labelled space (see Theorem 6.4). Then in Section 7 we show how our K -theory formulas reduce to those for graph algebras, ultragraph algebras and Matsumoto algebras in the appropriate cases. Furthermore in Proposition 7.6 we realise the Carlsen algebra $C^*(X)$ associated to a one-sided shift space X as a labelled graph algebra and hence compute its K -theory (see Corollary 7.7). In Section 8 we show how to compute the K -theory of certain labelled graph algebras as inductive limits; this is done in analogy with the computations of [35]. Finally, in Section 9 we provide a few examples of the computations outlined in Section 8.

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2. LABELLED SPACES AND THEIR C^* -ALGEBRAS

We will in this section briefly review the definitions of labelled spaces and their C^* -algebras.

Directed graphs. A directed graph E consists of a quadruple (E^0, E^1, r, s) where E^0 and E^1 are sets of vertices and edges respectively and $r, s : E^1 \rightarrow E^0$ are maps giving the direction of each edge. A path $\lambda = \lambda_1 \dots \lambda_n$ is a sequence of edges $\lambda_i \in E^1$ such that $r(\lambda_i) = s(\lambda_{i+1})$ for $i = 1, \dots, n-1$, we define $s(\lambda) = s(\lambda_1)$ and $r(\lambda) = r(\lambda_n)$. The collection of paths of length n in E is denoted by E^n and the collection of all finite paths in E by E^* , so that $E^* = \bigcup_{n \geq 1} E^n$.

A *loop* in E is a path which begins and ends at the same vertex, that is $\lambda \in E^*$ with $s(\lambda) = r(\lambda)$. We say that E is *row-finite* if every vertex emits finitely many edges. We denote the collection of all infinite paths in E by E^∞ . A vertex $v \in E^0$ is a *sink* if $s^{-1}(v) = \emptyset$ and we define E_{sink}^0 to be the set of all sinks in E^0 .

Labelled graphs. A *labelled graph* (E, \mathcal{L}) over a countable alphabet \mathcal{A} consists of a directed graph E together with a labelling map $\mathcal{L} : E^1 \rightarrow \mathcal{A}$. By replacing \mathcal{A} with $\mathcal{L}(E^1)$ (if necessary) we may assume that the map \mathcal{L} is onto.

Let \mathcal{A}^* be the collection of all *words* in the symbols of \mathcal{A} . The map \mathcal{L} extends naturally to a map $\mathcal{L} : E^n \rightarrow \mathcal{A}^*$, where $n \geq 1$. For $\lambda = e_1 \dots e_n \in E^n$ we set $\mathcal{L}(\lambda) = \mathcal{L}(e_1) \dots \mathcal{L}(e_n)$. In this case the path $\lambda \in E^n$ is said to be a *representative* of the *labelled path* $\mathcal{L}(e_1) \dots \mathcal{L}(e_n)$. Let $\mathcal{L}(E^n)$ denote the collection of all labelled paths in (E, \mathcal{L}) of length n where we write $|\alpha| = n$ if

$\alpha \in \mathcal{L}(E^n)$. The set $\mathcal{L}^*(E) = \bigcup_{n \geq 1} \mathcal{L}(E^n)$ is the collection of all labelled paths in the labelled graph (E, \mathcal{L}) .

The labelled graph (E, \mathcal{L}) is *left-resolving* if for all $v \in E^0$ the map $\mathcal{L} : r^{-1}(v) \rightarrow \mathcal{A}$ is injective. The left-resolving condition ensures that for all $v \in E^0$ the labels of the incoming edges to v are all different. For $\alpha \in \mathcal{L}^*(E)$ we put

$$s_{\mathcal{L}}(\alpha) = \{s(\lambda) \in E^0 : \mathcal{L}(\lambda) = \alpha\} \text{ and } r_{\mathcal{L}}(\alpha) = \{r(\lambda) \in E^0 : \mathcal{L}(\lambda) = \alpha\},$$

so that $r_{\mathcal{L}}, s_{\mathcal{L}} : \mathcal{L}^*(E) \rightarrow 2^{E^0}$, where 2^{E^0} denotes the set of subsets of E^0 . We shall drop the subscript on $r_{\mathcal{L}}$ and $s_{\mathcal{L}}$ if the context in which it is being used is clear.

For $A \subseteq E^0$ and $\alpha \in \mathcal{L}^*(E)$ the *relative range of α with respect to A* is defined to be

$$r_{\mathcal{L}}(A, \alpha) = \{r(\lambda) : \lambda \in E^*, \mathcal{L}(\lambda) = \alpha, s(\lambda) \in A\}.$$

Lemma 2.1. *Let (E, \mathcal{L}) be a left-resolving labelled graph. Then for $A, B \in 2^{E^0}$ with $B \subseteq A$ and $\beta \in \mathcal{L}^*(E)$ we have*

$$(1) \quad r(A \setminus B, \beta) = r(A, \beta) \setminus r(B, \beta).$$

Proof. Let $v \in r(A \setminus B, \beta)$. Then v receives a path μ labelled β from a vertex in A which is not in B . Since (E, \mathcal{L}) is left-resolving it follows that v cannot receive another path labelled β and so $v \in r(A, \beta) \setminus r(B, \beta)$.

Now suppose that $v \in r(A, \beta) \setminus r(B, \beta)$. Then v receives a path labelled β from a vertex in A and does not receive a path labelled β from any vertex in B . Hence $v \in r(A \setminus B, \beta)$. \square

Let (E, \mathcal{L}) be a labelled graph. A collection $\mathcal{B} \subseteq 2^{E^0}$ of subsets of E^0 is said to be *closed under relative ranges for (E, \mathcal{L})* if for all $A \in \mathcal{B}$ and $\alpha \in \mathcal{L}^*(E)$ we have $r(A, \alpha) \in \mathcal{B}$. If \mathcal{B} is closed under relative ranges for (E, \mathcal{L}) , contains $r(\alpha)$ for all $\alpha \in \mathcal{L}^*(E)$ and is also closed under finite intersections and unions, then we say that \mathcal{B} is *accommodating* for (E, \mathcal{L}) .

We are particularly interested in the sets of vertices which are the ranges of words in $\mathcal{L}^*(E)$, so we form

$$\mathcal{E}^- = \{\{v\} : v \in E_{\text{sink}}^0\} \cup \{r(\alpha) : \alpha \in \mathcal{L}^*(E)\}.$$

We then define $\mathcal{E}^{0,-}$ to be the smallest subset of 2^{E^0} which contains \mathcal{E}^- and is accommodating for (E, \mathcal{L}) . Of course, 2^{E^0} is the largest accommodating collection of subsets for (E, \mathcal{L}) .

Labelled spaces. A *labelled space* consists of a triple $(E, \mathcal{L}, \mathcal{B})$ where (E, \mathcal{L}) is a labelled graph and \mathcal{B} is accommodating for (E, \mathcal{L}) .

A labelled space $(E, \mathcal{L}, \mathcal{B})$ is *weakly left-resolving* if for every $A, B \in \mathcal{B}$ and every $\alpha \in \mathcal{L}^*(E)$ we have $r(A, \alpha) \cap r(B, \alpha) = r(A \cap B, \alpha)$. If (E, \mathcal{L}) is left-resolving then $(E, \mathcal{L}, \mathcal{B})$ is weakly left-resolving for any accommodating $\mathcal{B} \subseteq 2^{E^0}$.

Let (E, \mathcal{L}) be a labelled graph. For $A \subseteq E^0$ let $\mathcal{L}(AE^1) = \{\mathcal{L}(e) : e \in E^1, s(e) \in A\}$ (the set $\mathcal{L}(AE^1)$ was denoted L_A^1 in [4]).

Definition 2.2. Let $(E, \mathcal{L}, \mathcal{B})$ be a labelled space. If $\mathcal{L}(AE^1)$ is finite for all $A \in \mathcal{B}$, then we say that $(E, \mathcal{L}, \mathcal{B})$ is *set-finite*. If for all $A \in \mathcal{B}$ and all $\ell \geq 1$ the set $\{\mathcal{L}(\lambda) : \lambda \in E^\ell, r(\lambda) \in A\}$ is finite, then we say that $(E, \mathcal{L}, \mathcal{B})$ is *receiver set-finite*.

C*-algebras of labelled spaces. We recall from [4] the definition of the C*-algebra associated with the labelled space $(E, \mathcal{L}, \mathcal{B})$.

Definition 2.3. Let $(E, \mathcal{L}, \mathcal{B})$ be a weakly left-resolving labelled space. A representation of $(E, \mathcal{L}, \mathcal{B})$ in a C*-algebra consists of projections $\{p_A : A \in \mathcal{B}\}$ and partial isometries $\{s_a : a \in \mathcal{A}\}$ with the properties that:

- (i) If $A, B \in \mathcal{B}$, then $p_A p_B = p_{A \cap B}$ and $p_{A \cup B} = p_A + p_B - p_{A \cap B}$, where $p_\emptyset = 0$.
- (ii) If $a \in \mathcal{A}$ and $A \in \mathcal{B}$, then $p_A s_a = s_a p_{r(A, a)}$.

- (iii) If $a, b \in \mathcal{A}$, then $s_a^* s_a = p_{r(a)}$ and $s_a^* s_b = 0$ unless $a = b$.
- (iv) For $A \in \mathcal{B}$ with $\mathcal{L}(AE^1)$ finite and $A \cap E_{\text{sink}}^0 = \emptyset$ we have

$$(2) \quad p_A = \sum_{a \in \mathcal{L}(AE^1)} s_a p_{r(A,a)} s_a^*.$$

Remark 2.4. Notice that if the directed graph E contains sinks, then condition (iv) is different from condition (iv) of the original definition [4, Definition 4.1]. The original definition [4, Definition 4.1] was in error since it would lead to degeneracy of the vertex projections for sinks.

Notice also that if A contains a finite number of sinks and $\mathcal{B} = \mathcal{E}^{0,-}$ then we obtain the relation

$$p_A = \sum_{a \in \mathcal{L}(AE^1)} s_a p_{r(A,a)} s_a^* + \sum_{v \in A \cap E_{\text{sink}}^0} p_v.$$

Definition 2.5. Let $(E, \mathcal{L}, \mathcal{B})$ be a weakly left-resolving labelled space. Then $C^*(E, \mathcal{L}, \mathcal{B})$ is the C^* -algebra generated by a universal representation of $(E, \mathcal{L}, \mathcal{B})$.

The existence of $C^*(E, \mathcal{L}, \mathcal{B})$ is shown in [4, Theorem 4.5]. The universal property of $C^*(E, \mathcal{L}, \mathcal{B})$ allows us to define a strongly continuous action γ of \mathbf{T} on $C^*(E, \mathcal{L}, \mathcal{B})$ called the *gauge action* (see [4, Section 5]).

If $(E, \mathcal{L}, \mathcal{B})$ is a labelled space, then we let $\mathcal{L}^\#(E) = \mathcal{L}^*(E) \cup \{\epsilon\}$ where ϵ is a symbol not belonging to $\mathcal{L}^*(E)$ (ϵ denotes the empty word), and we let s_ϵ denote the unit of the multiplier algebra of $C^*(E, \mathcal{L}, \mathcal{B})$. It then follows from [4, Lemma 4.4] that we have

$$C^*(E, \mathcal{L}, \mathcal{B}) = \overline{\text{span}}\{s_\alpha p_A s_\beta^* : \alpha, \beta \in \mathcal{L}^\#(E), A \in \mathcal{B}\}.$$

3. PRELIMINARIES

We will in this section introduce some notation and presents some results which we will need in the rest of the paper.

Let $(E, \mathcal{L}, \mathcal{B})$ be a labelled space. We let $\hat{\mathcal{B}} = \{A \setminus B : A \in \mathcal{B}, B \in \mathcal{B} \cup \{\emptyset\}, B \subseteq A\}$ and let $\tilde{\mathcal{B}}$ denote the collection of subsets of E^0 which can be written as a finite disjoint union of sets belonging to $\hat{\mathcal{B}}$. It is straightforward to check that $\tilde{\mathcal{B}}$ is an algebra of subsets of E^0 (that is $\tilde{\mathcal{B}}$ is closed under relative complements and under finite intersections and unions). In fact, $\tilde{\mathcal{B}}$ is the smallest algebra containing \mathcal{B} .

Whenever $A \in 2^{E^0}$ we let χ_A denote the function defined on E^0 by

$$\chi_A(v) = \begin{cases} 1 & \text{if } v \in A, \\ 0 & \text{if } v \notin A. \end{cases}$$

We will regard χ_A as an element of the C^* -algebra of bounded functions on E^0 (when we deal with K -theory we will regard χ_A as an element of the group of functions from E^0 to \mathbb{Z}). Let $\mathcal{A}(E, \mathcal{L}, \mathcal{B})$ denote the C^* -subalgebra of the C^* -algebra of bounded functions on E^0 generated by $\{\chi_A : A \in \mathcal{B}\}$. It is easy to see that

$$\overline{\text{span}}\{\chi_A : A \in \tilde{\mathcal{B}}\} = \overline{\text{span}}\{\chi_A : A \in \hat{\mathcal{B}}\} = \overline{\text{span}}\{\chi_A : A \in \mathcal{B}\} = \mathcal{A}(E, \mathcal{L}, \mathcal{B}).$$

Lemma 3.1. *Let $(E, \mathcal{L}, \mathcal{B})$ be a labelled space and let $\{p_A : A \in \mathcal{B}\}$ be a family of projections in a C^* -algebra \mathcal{X} such that $p_{A \cap B} = p_A p_B$, $p_{A \cup B} = p_A + p_B - p_{A \cap B}$ for all $A, B \in \mathcal{B}$ and $p_\emptyset = 0$. Then there is a unique $*$ -homomorphism $\phi : \mathcal{A}(E, \mathcal{L}, \mathcal{B}) \rightarrow \mathcal{X}$ such that $\phi(\chi_A) = p_A$ for all $A \in \mathcal{B}$.*

Proof. Since $\mathcal{A}(E, \mathcal{L}, \mathcal{B})$ is generated by $\{\chi_A : A \in \mathcal{B}\}$, there can be at most one $*$ -homomorphism $\phi : \mathcal{A}(E, \mathcal{L}, \mathcal{B}) \rightarrow \mathcal{X}$ satisfying $\phi(\chi_A) = p_A$ for all $A \in \mathcal{B}$. We show that such a $*$ -homomorphism exists.

It is not difficult to show that if $D \in \hat{\mathcal{B}}$, then $p_A - p_B$ does not depend on the particular choice of A and B as long as $A \in \mathcal{B}$, $B \in \mathcal{B} \cup \{\emptyset\}$, $B \subseteq A$ and $D = B \setminus A$. We will in that case denote $p_A - p_B$ by p_D .

It is straightforward to check that if $D_1, D_2 \in \hat{\mathcal{B}}$, then $D_1 \cap D_2 \in \hat{\mathcal{B}}$ and $p_{D_1} p_{D_2} = p_{D_1 \cap D_2}$. It is also straightforward to check that if D_1, D_2, \dots, D_n are disjoint elements of $\hat{\mathcal{B}}$ and $\bigcup_{i=1}^n D_i \in \hat{\mathcal{B}}$, then $\sum_{i=1}^n p_{D_i} = p_{\bigcup_{i=1}^n D_i}$. It follows that if $B \in \tilde{\mathcal{B}}$, then $\sum_{i=1}^n p_{D_i}$ does not depend of the choice of D_i for $i = 1, \dots, n$, as long as D_1, D_2, \dots, D_n are disjoint elements belonging to $\hat{\mathcal{B}}$ such that $B = \bigcup_{i=1}^n D_i$. We will in that case denote $\sum_{i=1}^n p_{D_i}$ by p_B .

Let \mathfrak{F} denote the collection of finite subsets of $\tilde{\mathcal{B}}$ which are closed under relative complements, intersections and unions. Then $\bigcup_{B' \in \mathfrak{F}} \text{span}\{\chi_A : A \in B'\}$ is dense in $\mathcal{A}(E, \mathcal{L}, \mathcal{B})$ and so it is enough to prove that for every $B' \in \mathfrak{F}$ there is a $*$ -homomorphism $\phi_{B'} : \text{span}\{\chi_A : A \in B'\} \rightarrow \mathcal{X}$ satisfying $\phi_{B'}(\chi_A) = p_A$ for every $A \in B'$.

If $B' \in \mathfrak{F}$, then, since B' is closed under relative complements, intersections and unions, there is a collection F of mutually disjoint elements of B' such that $\text{span}\{\chi_A : A \in B'\} = \text{span}\{\chi_A : A \in F\}$. It follows that the map $\phi_{B'} : \text{span}\{\chi_A : A \in B'\} \rightarrow \mathcal{X}$ defined by

$$\phi_{B'} \left(\sum_{A \in F} c_A \chi_A \right) = \sum_{A \in F} c_A p_A$$

is a well-defined $*$ -homomorphism which maps χ_A to p_A for every $A \in B'$. Our result follows. \square

Proposition 3.2. *Let (E, \mathcal{L}) be a left-resolving labelled graph and let \mathcal{B} be an accommodating set for (E, \mathcal{L}) . Then $\tilde{\mathcal{B}}$ is accommodating for (E, \mathcal{L}) and $C^*(E, \mathcal{L}, \mathcal{B}) \cong C^*(E, \mathcal{L}, \tilde{\mathcal{B}})$.*

Proof. It follows from Lemma 2.1 that $\tilde{\mathcal{B}}$ is closed under relative ranges, and it contains $r(\alpha)$ for all $\alpha \in \mathcal{L}^*(E^1)$ since $\mathcal{B} \subseteq \tilde{\mathcal{B}}$. Since $\tilde{\mathcal{B}}$ is also closed under finite intersections and unions, it follows that $\tilde{\mathcal{B}}$ is accommodating for (E, \mathcal{L}) .

Let $\{s_a, p_B : a \in \mathcal{A}, B \in \mathcal{B}\}$ be a universal representation of $(E, \mathcal{L}, \mathcal{B})$ and $\{t_a, q_B : a \in \mathcal{A}, B \in \tilde{\mathcal{B}}\}$ a universal representation of $(E, \mathcal{L}, \tilde{\mathcal{B}})$. Clearly $\{t_a, q_B : a \in \mathcal{A}, B \in \mathcal{B}\}$ is a representation of $(E, \mathcal{L}, \mathcal{B})$. The universal property of $\{s_a, p_B : a \in \mathcal{A}, B \in \mathcal{B}\}$ then gives us a $*$ -homomorphism $\pi_{t,q} : C^*(E, \mathcal{L}, \mathcal{B}) \rightarrow C^*(E, \mathcal{L}, \tilde{\mathcal{B}})$ satisfying $\pi_{t,q}(s_a) = t_a$ for $a \in \mathcal{A}$ and $\pi_{t,q}(p_B) = q_B$ for $B \in \mathcal{B}$.

According to Lemma 3.1, there is a $*$ -homomorphism $\phi : \mathcal{A}(E, \mathcal{L}, \mathcal{B}) \rightarrow C^*(E, \mathcal{L}, \mathcal{B})$ such that $\phi(\chi_A) = p_A$ for all $A \in \mathcal{B}$. Recall that $\overline{\text{span}}\{\chi_A : A \in \tilde{\mathcal{B}}\} = \mathcal{A}(E, \mathcal{L}, \mathcal{B})$. We let $p_B = \phi(\chi_B)$ for $B \in \tilde{\mathcal{B}}$. One may then use Lemma 2.1 to show that $\{s_a, p_B : a \in \mathcal{A}, B \in \tilde{\mathcal{B}}\}$ is a representation of $(E, \mathcal{L}, \tilde{\mathcal{B}})$. The universal property of $\{t_a, q_B : a \in \mathcal{A}, B \in \tilde{\mathcal{B}}\}$ then gives us a $*$ -homomorphism $\pi_{s,p} : C^*(E, \mathcal{L}, \tilde{\mathcal{B}}) \rightarrow C^*(E, \mathcal{L}, \mathcal{B})$ satisfying $\pi_{s,p}(t_a) = s_a$ for $a \in \mathcal{A}$ and $\pi_{s,p}(q_B) = p_B$ for $B \in \tilde{\mathcal{B}}$.

The result follows as $\pi_{s,p} \circ \pi_{t,q}$ is the identity map on $C^*(E, \mathcal{L}, \mathcal{B})$ and $\pi_{t,q} \circ \pi_{s,p}$ is the identity map on $C^*(E, \mathcal{L}, \tilde{\mathcal{B}})$. \square

Lemma 3.3. *Let $(E, \mathcal{L}, \mathcal{B})$ be a labelled space and let \mathcal{I} be a closed two-sided ideal of $\mathcal{A}(E, \mathcal{L}, \mathcal{B})$. Then we have*

$$\mathcal{I} = \overline{\text{span}}\{\chi_A : A \in \hat{\mathcal{B}}, \chi_A \in \mathcal{I}\}.$$

Proof. We certainly have $\overline{\text{span}}\{\chi_A : A \in \hat{\mathcal{B}}, \chi_A \in \mathcal{I}\} \subseteq \mathcal{I}$. To prove the reverse inclusion, suppose that $f \in \mathcal{I}$ and let $\epsilon > 0$. Then there is a finite collection F of mutually disjoint

elements of $\hat{\mathcal{B}}$ and a family $(c_A)_{A \in F}$ of complex numbers such that

$$\left\| f - \sum_{A \in F} c_A \chi_A \right\| < \frac{\varepsilon}{2}.$$

Let $q_{\mathcal{I}} : \mathcal{A}(E, \mathcal{L}, \mathcal{B}) \rightarrow \mathcal{A}(E, \mathcal{L}, \mathcal{B})/\mathcal{I}$ be the quotient map. Then

$$\left\| \sum_{A \in F, \chi_A \notin \mathcal{I}} c_A q_{\mathcal{I}}(\chi_A) \right\| = \left\| q_{\mathcal{I}} \left(f - \sum_{A \in F} c_A \chi_A \right) \right\| < \frac{\varepsilon}{2},$$

and since $(q_{\mathcal{I}}(\chi_A))_{A \in F, \chi_A \notin \mathcal{I}}$ is a family of mutually orthogonal non-zero projections we must have $|c_A| < \frac{\varepsilon}{2}$ for all $A \in F$ such that $\chi_A \notin \mathcal{I}$. It then follows that

$$\left\| f - \sum_{A \in F, \chi_A \in \mathcal{I}} c_A \chi_A \right\| < \varepsilon$$

from which we may deduce that $f \in \overline{\text{span}}\{\chi_A : A \in \hat{\mathcal{B}}, \chi_A \in \mathcal{I}\}$. Our result follows. \square

4. REGULAR LABELLED SPACES

We will in this section introduce *regular* labelled spaces which are the labelled spaces for which we will show that the associated C^* -algebra can be constructed as a Cuntz-Pimsner algebra.

Definition 4.1. Let $(E, \mathcal{L}, \mathcal{B})$ be a weakly left-resolving labelled space. We say that $(E, \mathcal{L}, \mathcal{B})$ is *regular* if $A = B$ whenever $A, B \in \mathcal{B}$, $\mathcal{L}(AE^1) = \mathcal{L}(BE^1)$ is finite, $A \cap E_{\text{sink}}^0 = B \cap E_{\text{sink}}^0 = \emptyset$ and $r(A, a) = r(B, a)$ for all $a \in \mathcal{L}(AE^1) = \mathcal{L}(BE^1)$.

Remark 4.2. Let $(E, \mathcal{L}, \mathcal{B})$ be a weakly left-resolving labelled space. It follows directly from (iv) of Definition 2.3 that a necessary condition for the existence of an injective $*$ -homomorphism $\iota_{\mathcal{A}(E, \mathcal{L}, \mathcal{B})} : \mathcal{A}(E, \mathcal{L}, \mathcal{B}) \rightarrow C^*(E, \mathcal{L}, \mathcal{B})$ such that $\iota_{\mathcal{A}(E, \mathcal{L}, \mathcal{B})}(\chi_A) = p_A$ for every $A \in \mathcal{B}$, is that $(E, \mathcal{L}, \mathcal{B})$ is regular. We will in Corollary 5.7 see that this condition is also sufficient.

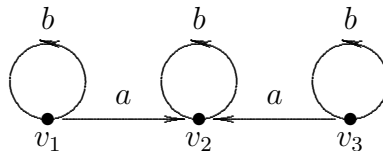
In [18, Example 2.4] there is an example of a labelled graph (E, \mathcal{L}) for which $(E, \mathcal{L}, \mathcal{E}^{0, -})$ is weakly left-resolving, but not regular. The following lemma and remark will provide us with lots of examples of regular weakly left-resolving labelled spaces.

Lemma 4.3. *Let $(E, \mathcal{L}, \mathcal{B})$ be a weakly left-resolving labelled space. If $r(A \setminus B, a) \subseteq r(A, a) \setminus r(B, a)$ for all $A, B \in \mathcal{B}$ and all $a \in \mathcal{A}$, then $(E, \mathcal{L}, \mathcal{B})$ is regular.*

Proof. Let $A, B \in \mathcal{B}$ and assume that $\mathcal{L}(AE^1) = \mathcal{L}(BE^1)$ is finite, $A \cap E_{\text{sink}}^0 = \emptyset$, $B \cap E_{\text{sink}}^0 = \emptyset$ and $r(A, a) = r(B, a)$ for all $a \in \mathcal{L}(AE^1) = \mathcal{L}(BE^1)$. Suppose, for contradiction, that $A \neq B$. Without loss of generality, we can assume that $A \setminus B \neq \emptyset$. Let $v \in A \setminus B$. Since $A \cap E_{\text{sink}}^0 = \emptyset$, it follows that there is an $e \in E^1$ with $s(e) = v$. Let $a = \mathcal{L}(e)$. Then $r(e) \in r(A \setminus B, a)$, but that contradicts the fact that $r(A \setminus B, a) \subseteq r(A, a) \setminus r(B, a) = \emptyset$. Thus, it must be the case that $A = B$. This shows that $(E, \mathcal{L}, \mathcal{B})$ is regular. \square

The following example shows that the converse of Lemma 4.3 does not hold in general.

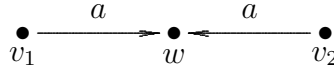
Example 4.4. Consider the labelled graph (E, \mathcal{L}) shown below.



Let $\mathcal{B} = \{\emptyset, \{v_2\}, \{v_3\}, \{v_1, v_3\}, \{v_2, v_3\}, \{v_1, v_2, v_3\}\}$. It is straightforward to check that \mathcal{B} is accommodating for (E, \mathcal{L}) and that the labelled space $(E, \mathcal{L}, \mathcal{B})$ is weakly left-resolving and regular. However $r(\{v_1, v_3\} \setminus \{v_3\}, a) = r(\{v_1\}, a) = \{v_2\}$ whereas $r(\{v_1, v_3\}, a) \setminus r(\{v_3\}, a) = \{v_2\} \setminus \{v_2\} = \emptyset$.

Remark 4.5. It follows from Lemma 2.1 and Lemma 4.3 that $(E, \mathcal{L}, \mathcal{B})$ is regular if (E, \mathcal{L}) is left-resolving labelled graph. The following example shows that the converse is not true in general.

Examples 4.6. Consider the labelled graph (E, \mathcal{L}) shown below.



This labelled graph is clearly not left-resolving. It is straightforward to check that the collection $\mathcal{B} = \{\{w\}, \emptyset\}$ is accommodating for (E, \mathcal{L}) and that the labelled space $(E, \mathcal{L}, \mathcal{B})$ is weakly left-resolving and regular.

5. VIEWING LABELLED GRAPH C^* -ALGEBRAS AS C^* -ALGEBRAS OF C^* -CORRESPONDENCES

Let $(E, \mathcal{L}, \mathcal{B})$ be a labelled space which is weakly left-resolving and regular. We will in this section show how to construct a C^* -correspondence $X(E, \mathcal{L}, \mathcal{B})$ whose Cuntz-Pimsner algebra (see [19] and [36]) is isomorphic to $C^*(E, \mathcal{L}, \mathcal{B})$.

For each $a \in \mathcal{A}$, let X_a be the ideal of $\mathcal{A}(E, \mathcal{L}, \mathcal{B})$ generated by $\chi_{r(a)}$ so that $f \in X_a$ if and only if $f(v) = 0$ for all $v \in E^0 \setminus r(a)$. Since X_a is an ideal, it is straightforward to see that X_a is a right Hilbert $\mathcal{A}(E, \mathcal{L}, \mathcal{B})$ -module with inner product defined by $\langle f, g \rangle = f^*g$ and right action given by the usual multiplication in $\mathcal{A}(E, \mathcal{L}, \mathcal{B})$.

We let $X(E, \mathcal{L}, \mathcal{B})$ be the right Hilbert $\mathcal{A}(E, \mathcal{L}, \mathcal{B})$ -module $\oplus_{a \in \mathcal{A}} X_a$. To turn $X(E, \mathcal{L}, \mathcal{B})$ into a C^* -correspondence we need to specify a left action of $\mathcal{A}(E, \mathcal{L}, \mathcal{B})$ on $X(E, \mathcal{L}, \mathcal{B})$, that is a $*$ -homomorphism $\phi : \mathcal{A}(E, \mathcal{L}, \mathcal{B}) \rightarrow \mathcal{L}(X(E, \mathcal{L}, \mathcal{B}))$ where $\mathcal{L}(X(E, \mathcal{L}, \mathcal{B}))$ denotes the C^* -algebra of adjointable operators on $X(E, \mathcal{L}, \mathcal{B})$ (see, for example, [19]).

Lemma 5.1. *For each $a \in \mathcal{A}$ there is a unique $*$ -homomorphism $\phi_a : \mathcal{A}(E, \mathcal{L}, \mathcal{B}) \rightarrow X_a$ satisfying $\phi_a(\chi_A) = \chi_{r(A, a)}$ for every $A \in \mathcal{B}$.*

Proof. This follows from Lemma 3.1 with $\mathcal{X} = X_a$ and $p_A = \chi_{r(A, a)}$ for $A \in \mathcal{B}$. □

Remark 5.2. If the labelled graph (E, \mathcal{L}) is left-resolving, then we have

$$\phi_a(f)(v) = \begin{cases} f(w) & \text{if } s(r^{-1}(v) \cap \mathcal{L}^{-1}(a)) = \{w\} \\ 0 & \text{if } s(r^{-1}(v) \cap \mathcal{L}^{-1}(a)) = \emptyset \end{cases}$$

for all $a \in \mathcal{A}$, $v \in E^0$ and $f \in \mathcal{A}(E, \mathcal{L}, \mathcal{B})$.

Lemma 5.3. *Let $a \in \mathcal{A}$ and $\phi_a : \mathcal{A}(E, \mathcal{L}, \mathcal{B}) \rightarrow X_a$ be as in Lemma 5.1. Then*

$$\ker(\phi_a) = \overline{\text{span}}\{\chi_{A \setminus B} : A \in \mathcal{B}, B \in \mathcal{B} \cup \{\emptyset\}, B \subseteq A, r(A, a) = r(B, a)\}.$$

Proof. This follows from Lemma 3.3 since for $A \in \mathcal{B}$ and $B \in \mathcal{B} \cup \{\emptyset\}$ with $B \subseteq A$, we have $\chi_{A \setminus B} \in \ker(\phi_a)$ if and only if $r(A, a) = r(B, a)$. □

Lemma 5.4. *Let $X(E, \mathcal{L}, \mathcal{B})$ be the right Hilbert $\mathcal{A}(E, \mathcal{L}, \mathcal{B})$ -module $\oplus_{a \in \mathcal{A}} X_a$.*

(i) For each $f \in \mathcal{A}(E, \mathcal{L}, \mathcal{B})$ the map

$$(3) \quad \varphi(f) : (x_a)_{a \in \mathcal{A}} \mapsto (\phi_a(f)x_a)_{a \in \mathcal{A}}$$

is an adjointable operator on $X(E, \mathcal{L}, \mathcal{B})$. The formula (3) defines a $*$ -homomorphism $\varphi : \mathcal{A}(E, \mathcal{L}, \mathcal{B}) \rightarrow \mathcal{L}(X(E, \mathcal{L}, \mathcal{B}))$. Hence $(X(E, \mathcal{L}, \mathcal{B}), \varphi)$ is a C^* -correspondence over $\mathcal{A}(E, \mathcal{L}, \mathcal{B})$.

(ii) For $a \in \mathcal{A}$ let $e_a = (\delta_{a,b}\chi_{r(a)})_{b \in \mathcal{A}} \in X(E, \mathcal{L}, \mathcal{B})$, where $\delta_{a,b}$ is the Kronecker delta function. Then we have

$$X(E, \mathcal{L}, \mathcal{B}) = \overline{\text{span}}_{\mathcal{A}(E, \mathcal{L}, \mathcal{B})} \{e_a : a \in \mathcal{A}\}.$$

Proof. Since the effect of $\varphi(f)$ is to multiply each term in $X(E, \mathcal{L}, \mathcal{B})$ by $\phi_a(f)$, one checks that (3) defines an adjointable operator with $\varphi(f)^* = \varphi(f^*)$. It follows that $f \mapsto \varphi(f)$ defines a $*$ -homomorphism φ from $\mathcal{A}(E, \mathcal{L}, \mathcal{B})$ to $\mathcal{L}(X(E, \mathcal{L}, \mathcal{B}))$.

The second statement holds since, for each $x = (x_a)_{a \in \mathcal{A}} \in X(E, \mathcal{L}, \mathcal{B})$ and each $\varepsilon > 0$, we have $\|x - \sum_{a \in F} e_a x_a\| < \varepsilon$ for some finite subset F of \mathcal{A} . \square

Let $\mathcal{K}(X(E, \mathcal{L}, \mathcal{B}))$ denote the ideal of $\mathcal{L}(X(E, \mathcal{L}, \mathcal{B}))$ consisting of generalized compact operators (see, for example, [19]). In order to define the Cuntz-Pimsner algebra $\mathcal{O}_{(X(E, \mathcal{L}, \mathcal{B}), \varphi)}$ associated with $(X(E, \mathcal{L}, \mathcal{B}), \varphi)$, we must first characterise the ideal

$$J_{(X(E, \mathcal{L}, \mathcal{B}), \varphi)} = \varphi^{-1}(\mathcal{K}(X(E, \mathcal{L}, \mathcal{B}))) \cap \ker(\varphi)^\perp$$

of $\mathcal{A}(E, \mathcal{L}, \mathcal{B})$ (cf. [19, Definition 3.2]). To do this, we let

$$\hat{\mathcal{B}}_J := \{A \in \hat{\mathcal{B}} : \mathcal{L}(AE^1) \text{ is finite and } A \cap B = \emptyset \text{ for all } B \in \mathcal{N}\}$$

where $\mathcal{N} = \{C \setminus D : C \in \mathcal{B}, D \in \mathcal{B} \cup \{\emptyset\}, D \subseteq C, r(C, a) = r(D, a) \text{ for all } a \in \mathcal{A}\}$.

Lemma 5.5. *The ideal $J_{(X(E, \mathcal{L}, \mathcal{B}), \varphi)}$ of $\mathcal{A}(E, \mathcal{L}, \mathcal{B})$ is given by*

$$J_{(X(E, \mathcal{L}, \mathcal{B}), \varphi)} = \overline{\text{span}}\{\chi_A : A \in \hat{\mathcal{B}}_J\}.$$

Proof. By Lemma 3.3 it is enough to prove that for $A \in \hat{\mathcal{B}}$, we have $\chi_A \in J_{(X(E, \mathcal{L}, \mathcal{B}), \varphi)}$ if and only if $A \in \hat{\mathcal{B}}_J$. Since $\ker(\varphi) = \bigcap_{a \in \mathcal{A}} \ker(\phi_a)$, it follows from Lemma 5.3 that

$$\ker(\varphi) = \overline{\text{span}}\{\chi_A : A \in \mathcal{N}\}$$

and hence we may deduce that

$$\ker(\varphi)^\perp = \overline{\text{span}}\{A \in \hat{\mathcal{B}} : A \cap B = \emptyset \text{ for all } B \in \mathcal{N}\}.$$

We claim that for each $\eta \in \mathcal{K}(X(E, \mathcal{L}, \mathcal{B}))$ the set $\{a \in \mathcal{A} : \|\langle e_a, \eta(e_a) \rangle\| \geq 1\}$ is finite. It suffices to check our claim for $\eta = \theta_{x,y}$ since these elements form a spanning set for $\mathcal{K}(X(E, \mathcal{L}, \mathcal{B}))$. For each $a_0 \in \mathcal{A}$ and $x = (x_a), y = (y_a) \in X(E, \mathcal{L}, \mathcal{B})$ one checks that $\langle e_{a_0}, \theta_{x,y} e_{a_0} \rangle = x_{a_0}^* y_{a_0}$, and since $\|x_{a_0}^* y_{a_0}\| \geq 1$ for only a finite number of a 's, our claim follows. So, if $A \in \hat{\mathcal{B}}$ and $\mathcal{L}(AE^1)$ is infinite, then $\varphi(\chi_A) \notin \mathcal{K}(X(E, \mathcal{L}, \mathcal{B}))$.

Conversely, if $A \in \hat{\mathcal{B}}$ and $\mathcal{L}(AE^1)$ is finite, then we have

$$(4) \quad \varphi(\chi_A) = \sum_{a \in \mathcal{L}(AE^1)} \theta_{e_a, e_a \chi_{r(A,a)}} \in \mathcal{K}(X(E, \mathcal{L}, \mathcal{B})).$$

Thus, if $A \in \hat{\mathcal{B}}$, then $\varphi(\chi_A) \in \mathcal{K}(X(E, \mathcal{L}, \mathcal{B}))$ if and only if $\mathcal{L}(AE^1)$ is finite. The result follows by definition of $J_{(X(E, \mathcal{L}, \mathcal{B}), \varphi)}$ and $\hat{\mathcal{B}}_J$. \square

Recall from [19, Definition 3.4] that a representation of the C^* -correspondence $(X(E, \mathcal{L}, \mathcal{B}), \varphi)$ on a C^* -algebra B consists of a pair (π, t) where $\pi : \mathcal{A}(E, \mathcal{L}, \mathcal{B}) \rightarrow B$ is a $*$ -homomorphism and $t : X(E, \mathcal{L}, \mathcal{B}) \rightarrow B$ is a linear map satisfying

- (1) $t(x)^*t(y) = \pi(\langle x, y \rangle)$ for $x, y \in X(E, \mathcal{L}, \mathcal{B})$ and
- (2) $\pi(f)t(x) = t(\varphi(f)x)$ for $f \in \mathcal{A}(E, \mathcal{L}, \mathcal{B})$ and $x \in X(E, \mathcal{L}, \mathcal{B})$.

Following [19, Definition 2.3] we define a $*$ -homomorphism $\psi_t : \mathcal{K}(X(E, \mathcal{L}, \mathcal{B})) \rightarrow B$ by

$$\psi_t(\theta_{x,y}) = t(x)t(y)^* \text{ for } x, y \in X(E, \mathcal{L}, \mathcal{B}).$$

Moreover, such a representation is called *covariant* if, in addition, we have $\pi(f) = \psi_t(\varphi(f))$ for all $f \in J_{(X(E, \mathcal{L}, \mathcal{B}), \varphi)}$. The *Cuntz-Pimsner algebra* $\mathcal{O}_{(X(E, \mathcal{L}, \mathcal{B}), \varphi)}$ of $(X(E, \mathcal{L}, \mathcal{B}), \varphi)$ is defined in [19] (see also [36]) to be the C^* -algebra generated by a universal covariant representations of $(X(E, \mathcal{L}, \mathcal{B}), \varphi)$.

Theorem 5.6. *Let $(E, \mathcal{L}, \mathcal{B})$ be a regular weakly left-resolving labelled space. Then there is a one-to-one correspondence between covariant representations of $(X(E, \mathcal{L}, \mathcal{B}), \varphi)$ and representations of $(E, \mathcal{L}, \mathcal{B})$ that takes a covariant representation (π, t) of $(X(E, \mathcal{L}, \mathcal{B}), \varphi)$ to the representation $\{\pi(\chi_A), t(e_a) : A \in \mathcal{B}, a \in \mathcal{A}\}$ of $(E, \mathcal{L}, \mathcal{B})$, and so $\mathcal{O}_{(X(E, \mathcal{L}, \mathcal{B}), \varphi)} \cong C^*(E, \mathcal{L}, \mathcal{B})$.*

Proof. Let $\{s_a, p_B : a \in \mathcal{A}, B \in \mathcal{B}\}$ be a universal representation of $(E, \mathcal{L}, \mathcal{B})$ and let (π, t) be a universal covariant representation of $(X(E, \mathcal{L}, \mathcal{B}), \varphi)$. It follows from Lemma 5.4 (ii) that $\mathcal{O}_{(X(E, \mathcal{L}, \mathcal{B}), \varphi)}$ is generated by $\{\pi(\chi_A), t(e_a) : A \in \mathcal{B}, a \in \mathcal{A}\}$. We claim that

$$\{\pi(\chi_A), t(e_a) : A \in \mathcal{B}, a \in \mathcal{A}\}$$

is a representation of $(E, \mathcal{L}, \mathcal{B})$.

It is straightforward to check that $\{\pi(\chi_A), t(e_a) : A \in \mathcal{B}, a \in \mathcal{A}\}$ satisfies (i)–(iii) of Definition 2.3. We will now show that $\{\pi(\chi_A), t(e_a) : A \in \mathcal{B}, a \in \mathcal{A}\}$ also satisfies (iv) of Definition 2.3. Assume $A \in \mathcal{B}$, $\mathcal{L}(AE^1)$ is finite and that $A \cap E_{\text{sink}}^0 = \emptyset$. Let $C \in \mathcal{B}$ and $D \in \mathcal{B} \cup \{\emptyset\}$ such that $D \subseteq C$ and $r(C, a) = r(D, a)$ for all $a \in \mathcal{A}$. Since $(E, \mathcal{L}, \mathcal{B})$ is weakly left-resolving, it follows that

$$r(A \cap C, a) = r(A, a) \cap r(C, a) = r(A, a) \cap r(D, a) = r(A \cap D, a)$$

for each $a \in \mathcal{A}$. It follows that $\mathcal{L}((A \cap C)E^1) = \mathcal{L}((A \cap D)E^1)$ (because $a \in \mathcal{L}(BE^1)$ if and only if $r(B, a) \neq \emptyset$). Since $\mathcal{L}((A \cap C)E^1) = \mathcal{L}((A \cap D)E^1) \subseteq \mathcal{L}(AE^1)$ is finite and $A \cap C \cap E_{\text{sink}}^0 = A \cap D \cap E_{\text{sink}}^0 = \emptyset$, it follows from the regularity of $(E, \mathcal{L}, \mathcal{B})$ that $A \cap C = A \cap D$. Thus $A \cap (C \setminus D) = \emptyset$. This shows that $A \in \hat{\mathcal{B}}_J$. It follows from Lemma 5.5 that $\chi_A \in J_{(X(E, \mathcal{L}, \mathcal{B}), \varphi)}$. By covariance and Equation (4) we have

$$\pi(\chi_A) = \psi_t(\phi(\chi_A)) = \psi_t\left(\sum_{a \in \mathcal{L}(AE^1)} \theta_{e_a, e_a \chi_{r(A, a)}}\right) = \sum_{a \in \mathcal{L}(AE^1)} t(e_a) \pi(\chi_{r(A, a)}) t(e_a)^*.$$

Thus $\{\pi(\chi_A), t(e_a) : A \in \mathcal{B}, a \in \mathcal{A}\}$ satisfies (iv) of Definition 2.3, and is therefore a representation of $(E, \mathcal{L}, \mathcal{B})$. It follows from the universality of $\{s_a, p_B : a \in \mathcal{A}, B \in \mathcal{B}\}$ that there exists a (unique) $*$ -homomorphism $\phi : C^*(E, \mathcal{L}, \mathcal{B}) \rightarrow \mathcal{O}_{(X(E, \mathcal{L}, \mathcal{B}), \varphi)}$ such that $\phi(s_a) = t(e_a)$ for $a \in \mathcal{A}$ and $\phi(p_A) = \pi(\chi_A)$ for $A \in \mathcal{B}$. This $*$ -homomorphism is surjective since $\mathcal{O}_{(X(E, \mathcal{L}, \mathcal{B}), \varphi)}$ is generated by $\{\pi(\chi_A), t(e_a) : A \in \mathcal{B}, a \in \mathcal{A}\}$.

It follows from the universality of (π, t) that there exists a strongly continuous action $\gamma : \mathbb{T} \rightarrow \text{Aut}(\mathcal{O}_{(X(E, \mathcal{L}, \mathcal{B}), \varphi)})$ such that $\gamma_z(t_{e_a}) = z t_{e_a}$ and $\gamma_z(\pi(\chi_A)) = \pi(\chi_A)$ for $z \in \mathbb{T}$, $a \in \mathcal{A}$ and $A \in \mathcal{B}$ (see [19, Page 377]). According to [19, Proposition 4.11], $\pi(\chi_A) \neq 0$ for $A \in \mathcal{B} \setminus \{\emptyset\}$. Thus it follows from [4, Theorem 5.3] that ϕ is injective and thus an isomorphism. The result then follows from the universality of $\{s_a, p_B : a \in \mathcal{A}, B \in \mathcal{B}\}$ and (π, t) . \square

Corollary 5.7. *Let $(E, \mathcal{L}, \mathcal{B})$ be a regular weakly left-resolving labelled space. Then there exists an injective $*$ -homomorphism $\iota_{\mathcal{A}(E, \mathcal{L}, \mathcal{B})} : \mathcal{A}(E, \mathcal{L}, \mathcal{B}) \rightarrow C^*(E, \mathcal{L}, \mathcal{B})$ such that $\iota_{\mathcal{A}(E, \mathcal{L}, \mathcal{B})}(\chi_A) = p_A$ for every $A \in \mathcal{B}$.*

Proof. Follows from [19, Proposition 4.11]. \square

Corollary 5.8. *Let $(E, \mathcal{L}, \mathcal{B})$ be a regular weakly left-resolving labelled space. Then $C^*(E, \mathcal{L}, \mathcal{B})$ is nuclear. If, in addition, \mathcal{B} and \mathcal{A} are countable, then $C^*(E, \mathcal{L}, \mathcal{B})$ satisfies the Universal Coefficient Theorem of [40].*

Proof. The first part follows from [19, Corollary 7.4] and the fact that $J_{(X(E, \mathcal{L}, \mathcal{B}), \varphi)}$ and $\mathcal{A}(E, \mathcal{L}, \mathcal{B})$ are commutative and thus nuclear. If, in addition, \mathcal{B} and \mathcal{A} are countable, then $\mathcal{A}(E, \mathcal{L}, \mathcal{B})$ and $(X(E, \mathcal{L}, \mathcal{B}), \varphi)$ are separable. It then follows from [19, Proposition 8.8] that $C^*(E, \mathcal{L}, \mathcal{B})$ satisfies the Universal Coefficient Theorem of [40]. \square

6. K-THEORY

We now compute the K -theory of $C^*(E, \mathcal{L}, \mathcal{B})$ when $(E, \mathcal{L}, \mathcal{B})$ is a regular weakly left-resolving labelled space. Our approach is to use [19, Theorem 8.6] which involves detailed knowledge of the map $[X] : K_*(J_{(X(E, \mathcal{L}, \mathcal{B}), \varphi)}) \rightarrow K_*(\mathcal{A}(E, \mathcal{L}, \mathcal{B}))$. Since we will need to work explicitly with this map, we will now recall its definition in detail (cf. [19, Definition 8.3]).

Let $D_{X(E, \mathcal{L}, \mathcal{B})}$ denote the linking algebra $\mathcal{K}(X(E, \mathcal{L}, \mathcal{B}) \oplus \mathcal{A}(E, \mathcal{L}, \mathcal{B}))$. Following [19, Definition B.1] we denote the natural embedding of $\mathcal{A}(E, \mathcal{L}, \mathcal{B})$ into $D_{X(E, \mathcal{L}, \mathcal{B})}$ by $\iota_{\mathcal{A}(E, \mathcal{L}, \mathcal{B})}$ and the natural embedding of $\mathcal{K}(X(E, \mathcal{L}, \mathcal{B}))$ into $D_{X(E, \mathcal{L}, \mathcal{B})}$ by $\iota_{\mathcal{K}(X(E, \mathcal{L}, \mathcal{B}))}$. It is shown in [19, Proposition B.3] that the map $(\iota_{\mathcal{A}(E, \mathcal{L}, \mathcal{B})})_* : K_*(\mathcal{A}(E, \mathcal{L}, \mathcal{B})) \rightarrow K_*(D_{X(E, \mathcal{L}, \mathcal{B})})$ induced by $\iota_{\mathcal{A}(E, \mathcal{L}, \mathcal{B})}$ is an isomorphism. The map $[X] : K_*(J_{(X(E, \mathcal{L}, \mathcal{B}), \varphi)}) \rightarrow K_*(\mathcal{A}(E, \mathcal{L}, \mathcal{B}))$ is then defined to be $(\iota_{\mathcal{A}(E, \mathcal{L}, \mathcal{B})})_*^{-1} \circ (\iota_{\mathcal{K}(X(E, \mathcal{L}, \mathcal{B}))})_* \circ (\varphi|_{J_{(X(E, \mathcal{L}, \mathcal{B}), \varphi)}})_*$.

Lemma 6.1. *Let $(E, \mathcal{L}, \mathcal{B})$ be a regular weakly left-resolving labelled space. Let $a \in \mathcal{A}$ and $A \in \hat{\mathcal{B}}$ such that $A \subseteq r(a)$. Then for $[\chi_A]_0 \in K_0(\mathcal{A}(E, \mathcal{L}, \mathcal{B}))$ and $[\theta_{e_a, e_a \chi_A}]_0 \in K_0(\mathcal{K}(X(E, \mathcal{L}, \mathcal{B})))$ we have*

$$(\iota_{\mathcal{A}(E, \mathcal{L}, \mathcal{B})})_*([\chi_A]_0) = (\iota_{\mathcal{K}(X(E, \mathcal{L}, \mathcal{B}))})_*([\theta_{e_a, e_a \chi_A}]_0)$$

in $K_0(D_{X(E, \mathcal{L}, \mathcal{B})})$.

Proof. Let v be the function that maps $(\eta, f) \in X(E, \mathcal{L}, \mathcal{B}) \times \mathcal{A}(E, \mathcal{L}, \mathcal{B})$ to $(e_a \chi_A f, 0) \in X(E, \mathcal{L}, \mathcal{B}) \times \mathcal{A}(E, \mathcal{L}, \mathcal{B})$. One checks that $v \in \mathcal{K}(X(E, \mathcal{L}, \mathcal{B}) \oplus \mathcal{A}(E, \mathcal{L}, \mathcal{B}))$, that $vv^* = \iota_{\mathcal{K}(X(E, \mathcal{L}, \mathcal{B}))}(\theta_{e_a, e_a \chi_A})$ and $v^*v = \iota_{\mathcal{A}(E, \mathcal{L}, \mathcal{B})}(\chi_A)$. It follows that the elements $\iota_{\mathcal{K}(X(E, \mathcal{L}, \mathcal{B}))}(\theta_{e_a, e_a \chi_A})$ and $\iota_{\mathcal{A}(E, \mathcal{L}, \mathcal{B})}(\chi_A)$ are equivalent in $K_0(D_{X(E, \mathcal{L}, \mathcal{B})})$. \square

Since $\mathcal{A}(E, \mathcal{L}, \mathcal{B}) = \overline{\text{span}}\{\chi_A : A \in \hat{\mathcal{B}}\}$ and $J_{(X(E, \mathcal{L}, \mathcal{B}), \varphi)} = \overline{\text{span}}\{\chi_A : A \in \hat{\mathcal{B}}_J\}$, it follows that there is an isomorphism from $\text{span}_{\mathbb{Z}}\{\chi_A : A \in \hat{\mathcal{B}}\}$ to $K_0(\mathcal{A}(E, \mathcal{L}, \mathcal{B}))$ which for each $A \in \hat{\mathcal{B}}$ takes χ_A to $[\chi_A]_0$, and an isomorphism from $\text{span}_{\mathbb{Z}}\{\chi_A : A \in \hat{\mathcal{B}}_J\}$ to $K_0(J_{(X(E, \mathcal{L}, \mathcal{B}), \varphi)})$ which for each $A \in \hat{\mathcal{B}}_J$ takes χ_A to $[\chi_A]_0$. We will simply identify $K_0(\mathcal{A}(E, \mathcal{L}, \mathcal{B}))$ with $\text{span}_{\mathbb{Z}}\{\chi_A : A \in \hat{\mathcal{B}}\}$, $K_0(J_{(X(E, \mathcal{L}, \mathcal{B}), \varphi)})$ with $\text{span}_{\mathbb{Z}}\{\chi_A : A \in \hat{\mathcal{B}}_J\}$ and each $[\chi_A]_0$ with χ_A .

Lemma 6.2. *Identifying $K_0(\mathcal{A}(E, \mathcal{L}, \mathcal{B}))$ with $\text{span}_{\mathbb{Z}}\{\chi_A : A \in \hat{\mathcal{B}}\}$ and $K_0(J_{(X(E, \mathcal{L}, \mathcal{B}), \varphi)})$ with $\text{span}_{\mathbb{Z}}\{\chi_A : A \in \hat{\mathcal{B}}_J\}$, the homomorphism $[X] : K_0(J_{(X(E, \mathcal{L}, \mathcal{B}), \varphi)}) \rightarrow K_0(\mathcal{A}(E, \mathcal{L}, \mathcal{B}))$ induces the map $\Phi : \text{span}_{\mathbb{Z}}\{\chi_A : A \in \hat{\mathcal{B}}_J\} \rightarrow \text{span}_{\mathbb{Z}}\{\chi_A : A \in \hat{\mathcal{B}}\}$ determined by*

$$(5) \quad \chi_A \mapsto \sum_{a \in \mathcal{L}(AE^1)} \chi_{r(A, a)} \text{ for } A \in \hat{\mathcal{B}}_J.$$

Proof. Let $A \in \hat{\mathcal{B}}_J$. Since $\phi_a(\chi_A) = \chi_{r(A,a)}$ is a projection in $\mathcal{A}(E, \mathcal{L}, \mathcal{B})$, we have $\varphi(\chi_A) = \sum_{a \in \mathcal{L}(AE^1)} \theta_{e_a, e_a \chi_{r(A,a)}}$ from Equation (4). It follows from Lemma 6.1 that

$$\begin{aligned} (\iota_{\mathcal{K}(X(E, \mathcal{L}, \mathcal{B}))})_* \circ (\varphi|_{J_{(X(E, \mathcal{L}, \mathcal{B}), \varphi)}})_*([\chi_A]_0) &= (\iota_{\mathcal{K}(X(E, \mathcal{L}, \mathcal{B}))})_* \left(\left[\sum_{a \in \mathcal{L}(AE^1)} \theta_{e_a, e_a \chi_{r(A,a)}} \right]_0 \right) \\ &= \sum_{a \in \mathcal{L}(AE^1)} (\iota_{\mathcal{K}(X(E, \mathcal{L}, \mathcal{B}))})_*([\theta_{e_a, e_a \chi_{r(A,a)}}]_0) \\ &= \sum_{a \in \mathcal{L}(AE^1)} (\iota_{\mathcal{A}(E, \mathcal{L}, \mathcal{B})})_*([\chi_{r(A,a)}]_0), \end{aligned}$$

from which Equation (5) follows using the given identifications and the definition of $[X]$. \square

If $\mathcal{B} \subseteq 2^{E^0}$ is uncountable, then $\mathcal{A}(E, \mathcal{L}, \mathcal{B})$ is not separable. However, it is still locally finite dimensional. Such nonseparable AF algebras were considered in [20].

Lemma 6.3. *Let $(E, \mathcal{L}, \mathcal{B})$ be a regular weakly left-resolving labelled space. Then $\mathcal{A}(E, \mathcal{L}, \mathcal{B})$ and $J_{(X(E, \mathcal{L}, \mathcal{B}), \varphi)}$ are locally finite dimensional algebras with $K_1(\mathcal{A}(E, \mathcal{L}, \mathcal{B})) = K_1(J_{(X(E, \mathcal{L}, \mathcal{B}), \varphi)}) = 0$.*

Proof. We prove the result for $\mathcal{A}(E, \mathcal{L}, \mathcal{B})$ and note that the proof for $J_{(X(E, \mathcal{L}, \mathcal{B}), \varphi)}$ is similar. The first statement follows by the argument given in the proof of [4, Theorem 5.3]. To prove that $K_1(\mathcal{A}(E, \mathcal{L}, \mathcal{B})) = 0$, we argue as follows¹. By adding E^0 to \mathcal{B} (if necessary) we may assume that $\mathcal{A}(E, \mathcal{L}, \mathcal{B})$ is unital. Let u be a unitary in $\mathcal{A}(E, \mathcal{L}, \mathcal{B})$, then given $0 < \epsilon < 1$ there is a unitary u' in $\text{span}_{\mathbb{Z}}\{\chi_A : A \in \hat{\mathcal{B}}\}$ such that $\|u - u'\| < \epsilon$. Using the local finite dimensionality of $\mathcal{A}(E, \mathcal{L}, \mathcal{B})$ one can show that u' belongs to a finite dimensional subalgebra U of $\mathcal{A}(E, \mathcal{L}, \mathcal{B})$ which contains the identity $1_{\mathcal{A}(E, \mathcal{L}, \mathcal{B})}$. Since the unitary group of a finite dimensional C^* -algebra is connected there is a continuous path of unitaries from u' to $1_{\mathcal{A}(E, \mathcal{L}, \mathcal{B})} = \chi_{E^0}$ which remains continuous when we consider U as a subalgebra of $\mathcal{A}(E, \mathcal{L}, \mathcal{B})$. Since $\|u - u'\| < \epsilon < 1$ there is a continuous path of unitaries from u to u' and hence from u to $1_{\mathcal{A}(E, \mathcal{L}, \mathcal{B})}$. The result now follows. \square

Theorem 6.4. *Let $(E, \mathcal{L}, \mathcal{B})$ be a regular weakly left-resolving labelled space. Let $(1 - \Phi)$ be the linear map from $\text{span}_{\mathbb{Z}}\{\chi_A : A \in \hat{\mathcal{B}}_J\}$ to $\text{span}_{\mathbb{Z}}\{\chi_A : A \in \hat{\mathcal{B}}\}$ given by*

$$(1 - \Phi)(\chi_A) = \chi_A - \sum_{a \in \mathcal{L}(AE^1)} \chi_{r(A,a)}, \quad A \in \hat{\mathcal{B}}_J.$$

Then $K_1(C^(E, \mathcal{L}, \mathcal{B})) \cong \ker(1 - \Phi)$, and $K_0(C^*(E, \mathcal{L}, \mathcal{B})) \cong \text{span}_{\mathbb{Z}}\{\chi_A : A \in \hat{\mathcal{B}}\} / \text{Im}(1 - \Phi)$ via $[p_A]_0 \mapsto \chi_A + \text{Im}(1 - \Phi)$ for $A \in \hat{\mathcal{B}}$.*

Proof. The result follows by [19, Theorem 8.6], Theorem 5.6, Lemma 6.2 and the fact that $K_1(\mathcal{A}(E, \mathcal{L}, \mathcal{B})) = K_1(J_{(X(E, \mathcal{L}, \mathcal{B}), \varphi)}) = 0$ from Lemma 6.3. \square

7. EXAMPLES

In this section we show that Theorem 6.4 unifies the K -theory formulas for several interesting classes of C^* -algebras.

¹Thanks to Iain Raeburn for showing us how to do this.

Example 1 – Graph algebras. Let E be a directed graph. Following [4, Examples 3.3 and 4.3 (i)] we may realise $C^*(E)$ as the C^* -algebra of a regular weakly left-resolving labelled space in the following manner. We take the trivial labelling $\mathcal{L}_t : E^1 \rightarrow E^1$ defined for $e \in E^1$ by $\mathcal{L}_t(e) = e$. Then (E, \mathcal{L}) is a left-resolving labelled graph. If we let \mathcal{B} be the set of all finite subsets of E^0 , then \mathcal{B} is accommodating. Since (E, \mathcal{L}) is left-resolving, it follows that the labelled space $(E, \mathcal{L}_t, \mathcal{B})$ is weakly left-resolving and regular (see Page 3 and Remark 4.5). By [4, Proposition 5.1 (i)] there is an isomorphism $\phi : C^*(E) \rightarrow C^*(E, \mathcal{L}_t, \mathcal{B})$ that maps P_v to $p_{\{v\}}$ and S_e to s_e where $\{S_e, P_v : e \in E^1, v \in E^0\}$ and $\{s_e, p_A : e \in E^1, A \in \mathcal{B}\}$ are the canonical generators of $C^*(E)$ and $C^*(E, \mathcal{L}_t, \mathcal{B})$ respectively.

Let $E_{ns}^0 = \{v \in E^0 : 0 < |s^{-1}(v)| < \infty\}$ denote the set of nonsingular vertices in E^0 . For each $v \in E_{ns}^0$ we let $e_v \in \bigoplus_{w \in E_{ns}^0} \mathbb{Z}$ be defined by $e_v = (\delta_{v,w})_{w \in E_{ns}^0}$. We similarly define $e_v \in \bigoplus_{v \in E^0} \mathbb{Z}$ for each $v \in E^0$.

Note that $\hat{\mathcal{B}}_J$ is precisely the collection of all finite subsets of E_{ns}^0 . One checks that there are isomorphisms $\psi : \text{span}_{\mathbb{Z}}\{\chi_A : A \in \hat{\mathcal{B}}\} \rightarrow \bigoplus_{v \in E^0} \mathbb{Z}$ and $\psi_J : \text{span}_{\mathbb{Z}}\{\chi_A : A \in \hat{\mathcal{B}}_J\} \rightarrow \bigoplus_{v \in E_{ns}^0} \mathbb{Z}$ such that for all $v \in E^0$ (respectively $v \in E_{ns}^0$) we have $\psi(\chi_v) = e_v$ (respectively $\psi_J(\chi_v) = e_v$). For $v \in E_{ns}^0$ we have $\Phi(\chi_v) = \sum_{f \in s^{-1}\{v\}} \chi_{r(f)}$ and we may now deduce the following Corollary from Theorem 6.4.

Corollary 7.1 (cf. [13, Theorem 3.1]). *Let E be a directed graph. With notation as above, define a linear map $(1 - \Phi) : \bigoplus_{v \in E_{ns}^0} \mathbb{Z} \rightarrow \bigoplus_{v \in E^0} \mathbb{Z}$ by*

$$(1 - \Phi)(e_v) = e_v - \sum_{f \in s^{-1}\{v\}} e_{r(f)}, \quad v \in E_{ns}^0.$$

Then $K_1(C^(E))$ is isomorphic to $\ker(1 - \Phi)$ and there exists an isomorphism from $K_0(C^*(E))$ to $\text{coker}(1 - \Phi)$ which maps $[P_v]_0$ to $e_v + \text{Im}(1 - \Phi)$ for each $v \in E^0$.*

Example 2 – Ultragraph algebras. For the definition and properties of an ultragraph see [37]. Following [4, Examples 3.3 and 4.3 (ii)] we may realise an ultragraph as a labelled graph as follows: Let $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ be an ultragraph. We define a directed graph $E_{\mathcal{G}} = (E_{\mathcal{G}}^0, E_{\mathcal{G}}^1, r', s')$ by putting $E_{\mathcal{G}}^0 = G^0$, $E_{\mathcal{G}}^1 = \{(e, w) : e \in \mathcal{G}^1, w \in r(e)\}$, $r'(e, w) = w$ and $s'(e, w) = s(e)$. Setting $\mathcal{A} = \mathcal{G}^1$ we may define a labelling map $\mathcal{L}_{\mathcal{G}} : E^1 \rightarrow \mathcal{A}$ by $\mathcal{L}_{\mathcal{G}}(e, w) = e$. If we set $\mathcal{B} = \mathcal{G}^0$ as defined in [37, Section 2], then $(E_{\mathcal{G}}, \mathcal{L}_{\mathcal{G}}, \mathcal{B})$ is a regular weakly left-resolving labelled space. By [4, Proposition 5.1 (ii)] there is an isomorphism $\phi : C^*(\mathcal{G}) \rightarrow C^*(E_{\mathcal{G}}, \mathcal{L}_{\mathcal{G}}, \mathcal{B})$ that maps P_A to p_A and S_e to s_e where $\{S_e, P_A\}$ and $\{s_e, p_A\}$ are the canonical generators of $C^*(\mathcal{G})$ and $C^*(E_{\mathcal{G}}, \mathcal{L}_{\mathcal{G}}, \mathcal{B})$ respectively.

The following definitions are taken from [21]. Let $\mathfrak{A}_{\mathcal{G}}$ denote the C^* -subalgebra of $\ell^\infty(G^0)$ generated by the point masses $\{\delta_v : v \in G^0\}$ and the characteristic functions $\{\chi_{r(e)} : e \in \mathcal{G}^1\}$ and let $Z_{\mathcal{G}}$ denote the algebraic subalgebra of $\ell^\infty(G^0, \mathbb{Z})$ generated by these functions. Let $G_{rg}^0 \subseteq G^0$ denote the set of vertices $v \in G^0$ satisfying $0 < |s^{-1}(v)| < \infty$. For $v \in G_{rg}^0$ let $e_v \in \bigoplus_{w \in G_{rg}^0} \mathbb{Z}$ be defined by $e_v = (\delta_{v,w})_{w \in G_{rg}^0}$ and define $e_v \in Z_{\mathcal{G}} \subseteq \ell^\infty(G^0, \mathbb{Z})$ similarly for $v \in E^0$.

It follows from [21, Proposition 2.6 and 2.20] that $Z_{\mathcal{G}} = K_0(\mathfrak{A}_{\mathcal{G}}) = \text{span}_{\mathbb{Z}}\{\chi_A : A \in \hat{\mathcal{B}}\}$. One checks that $A \in \hat{\mathcal{B}}$ belongs to $\hat{\mathcal{B}}_J$ if and only if A is a finite subset of G_{rg}^0 . It then follows that there is an isomorphism from $\bigoplus_{v \in G_{rg}^0} \mathbb{Z}$ to $\text{span}_{\mathbb{Z}}\{\chi_A : A \in \hat{\mathcal{B}}_J\}$ which maps e_v to $\chi_{\{v\}}$ for every $v \in G_{rg}^0$. If $v \in G_{rg}^0$, then $\Phi(\chi_{\{v\}}) = \sum_{f \in s^{-1}\{v\}} \chi_{r(f)}$, and we may deduce the following Corollary to Theorem 6.4.

Corollary 7.2 (cf. [21, Theorem 5.4]). *Let $\mathcal{G} = (G^0, \mathcal{G}^1, r, s)$ be an ultragraph. With notation as above, define a linear map $(1 - \Phi) : \bigoplus_{v \in G_{rg}^0} \mathbb{Z} \rightarrow Z_{\mathcal{G}}$ by*

$$(1 - \Phi)(e_v) = e_v - \sum_{f \in s^{-1}\{v\}} \chi_{r(f)}, \quad v \in G_{rg}^0.$$

Then $K_1(C^(\mathcal{G}))$ is isomorphic to $\ker(1 - \Phi)$ and there is an isomorphism from $K_0(C^*(\mathcal{G}))$ to $\text{coker}(1 - \Phi)$ which maps $[P_A]_0$ to $\chi_A + \text{Im}(1 - \Phi)$ for each $A \in \mathcal{G}^0$.*

Example 3 – Matsumoto algebras. We are interested in two C^* -algebras which Matsumoto associates to a (two-sided) shift space Λ over a finite alphabet \mathcal{A} . The first, which we denote \mathcal{O}_{Λ^*} , was described in [26], [27], [28], [29], [31] and [32]. It was shown in [4, Theorem 6.3] that $\mathcal{O}_{\Lambda^*} \cong C^*(E_{\Lambda^*}, \mathcal{L}_{\Lambda^*}, \mathcal{E}_{\Lambda^*}^{0,-})$ where $(E_{\Lambda^*}, \mathcal{L}_{\Lambda^*})$ is the predecessor graph (or past set cover) of Λ , as defined in [4, Examples 3.3] (vii) (see also [2, Definition 5.2]).

The second, which we denote \mathcal{O}_{Λ} was described in [8] and [34]. Under a technical hypothesis, analogous to Cuntz and Krieger's condition (I), it was shown in [4, Theorem 6.3] that $\mathcal{O}_{\Lambda} \cong C^*(E_{\Lambda}, \mathcal{L}_{\Lambda}, \mathcal{E}_{\Lambda}^{0,-})$ where $(E_{\Lambda}, \mathcal{L}_{\Lambda})$ is the left Krieger cover of Λ , as defined in [22, Section 2] (see also [4, Examples 3.3 (vi)] and [2, Definition 2.11]).

Let $\mathcal{A}^* \setminus \{\epsilon\}$ denote the set of non-empty finite words over \mathcal{A} . Let $\{S_a : a \in \mathcal{A}\}$ be the generators of \mathcal{O}_{Λ} (respectively \mathcal{O}_{Λ^*}). For $u = u_1 u_2 \dots u_m \in \mathcal{A}^* \setminus \{\epsilon\}$, let $S_u := S_{u_1} S_{u_2} \dots S_{u_m}$.

Corollary 7.3 (cf. [27, Theorem 4.9] and [33, Lemma 2.7]). *Let Λ be a (two-sided) shift space over a finite alphabet \mathcal{A} . Let $(1 - \Phi) : \text{span}_{\mathbb{Z}}\{\chi_A : A \in \mathcal{E}_{\Lambda^*}^{0,-}\} \rightarrow \text{span}_{\mathbb{Z}}\{\chi_A : A \in \mathcal{E}_{\Lambda}^{0,-}\}$ be the linear map given by*

$$(1 - \Phi)(\chi_A) = \chi_A - \sum_{a \in \mathcal{A}} \chi_{r(A,a)} \quad A \in \mathcal{E}_{\Lambda^*}^{0,-}.$$

Then $K_1(\mathcal{O}_{\Lambda^})$ is isomorphic to $\ker(1 - \Phi)$ and there exists an isomorphism from $K_0(\mathcal{O}_{\Lambda^*})$ to $\text{coker}(1 - \Phi)$ which maps $[S_u^* S_u]_0$ to $\chi_{r(u)} + \text{Im}(1 - \Phi)$ for each $u \in \mathcal{A}^* \setminus \{\epsilon\}$.*

If Λ satisfies condition (I) of [28], then we have similar formulas for $K_0(\mathcal{O}_{\Lambda})$ and $K_1(\mathcal{O}_{\Lambda})$.

Proof. The labelled spaces $(E_{\Lambda}, \mathcal{L}_{\Lambda}, \mathcal{E}_{\Lambda}^{0,-})$ and $(E_{\Lambda^*}, \mathcal{L}_{\Lambda^*}, \mathcal{E}_{\Lambda^*}^{0,-})$ are weakly left-resolving and regular since $(E_{\Lambda}, \mathcal{L}_{\Lambda})$ and $(E_{\Lambda^*}, \mathcal{L}_{\Lambda^*})$ are left-resolving. It is straightforward to check that $(\mathcal{E}_{\Lambda}^{0,-})^{\wedge} = (\mathcal{E}_{\Lambda}^{0,-})_J^{\wedge}$ and $(\mathcal{E}_{\Lambda^*}^{0,-})^{\wedge} = (\mathcal{E}_{\Lambda^*}^{0,-})_J^{\wedge}$. Since $\text{span}_{\mathbb{Z}}\{\chi_A : A \in \mathcal{E}_{\Lambda}^{0,-}\} = \text{span}_{\mathbb{Z}}\{\chi_A : A \in (\mathcal{E}_{\Lambda}^{0,-})^{\wedge}\}$ and $\text{span}_{\mathbb{Z}}\{\chi_A : A \in \mathcal{E}_{\Lambda^*}^{0,-}\} = \text{span}_{\mathbb{Z}}\{\chi_A : A \in (\mathcal{E}_{\Lambda^*}^{0,-})^{\wedge}\}$, the result follows from [4, Theorem 6.3] and Theorem 6.4. \square

Example 4 – Cuntz-Pimsner algebras associated with subshifts. Let \mathbf{X} be a one-sided shift space over a finite alphabet \mathcal{A} . In [7] a C^* -algebra was associated with \mathbf{X} . An alternative construction of this C^* -algebra is given in [9] and its relationship with the algebras \mathcal{O}_{Λ} and \mathcal{O}_{Λ^*} is explored. We will denote this C^* -algebra by $C^*(\mathbf{X})$ (it is denoted by $\mathcal{O}_{\mathbf{X}}$ in [7] and by $\mathcal{D}_{\mathbf{X}} \rtimes_{\alpha, \mathcal{L}} \mathbb{N}$ in [9]).

We show how one may realise $C^*(\mathbf{X})$ as the C^* -algebra of a labelled space. We let $E_{\mathbf{X}}$ be the directed graph $(E_{\mathbf{X}}^0, E_{\mathbf{X}}^1, r, s)$ where $E_{\mathbf{X}}^0 = \mathbf{X}$, $E_{\mathbf{X}}^1 = \{(x, a, y) \in \mathbf{X} \times \mathcal{A} \times \mathbf{X} : x = ay\}$ and $r, s : E_{\mathbf{X}}^1 \rightarrow E_{\mathbf{X}}^0$ are defined by $s(x, a, y) = x$ and $r(x, a, y) = y$. We let $\mathcal{L}_{\mathbf{X}} : E_{\mathbf{X}}^1 \rightarrow \mathcal{A}$ be the labeling given by $\mathcal{L}_{\mathbf{X}}(x, a, y) = a$.

For $u, v \in \mathcal{L}_{\mathbf{X}}^{\#}(E_{\mathbf{X}})$, let $C(u, v) = \{vx \in \mathbf{X} : ux \in \mathbf{X}\}$. Let $\mathcal{B}_{\mathbf{X}}$ be the Boolean algebra generated by $\{C(u, v) : u, v \in \mathcal{L}_{\mathbf{X}}^{\#}(E_{\mathbf{X}})\}$. That is, $\mathcal{B}_{\mathbf{X}}$ is the smallest subset of $2^{\mathbf{X}}$ which is closed under finite intersections, finite unions, relative complements and which contains $C(u, v)$ for every $u, v \in \mathcal{L}_{\mathbf{X}}^{\#}(E_{\mathbf{X}})$.

Lemma 7.4. *With the above definitions $(E_{\mathbf{X}}, \mathcal{L}_{\mathbf{X}}, \mathcal{B}_{\mathbf{X}})$ is a regular weakly left-resolving labelled space.*

Proof. We will first prove that \mathcal{B}_X is accommodating for (E_X, \mathcal{L}_X) . Let $\alpha \in \mathcal{L}_X^*(E_X)$. Then we have that $r(\alpha) = C(\alpha, \epsilon)$, from which it follows that \mathcal{B}_X contains $r(\alpha)$. It remains to show that \mathcal{B}_X is closed under relative ranges.

To check that \mathcal{B}_X is closed under relative ranges it suffices to check that $r(C(u, v), a) \in \mathcal{B}_X$ for all $u, v \in \mathcal{L}_X^\#(E_X)$ and $a \in \mathcal{A}$. If $u, v \in \mathcal{L}_X^\#(E_X)$ are such that $v = v_1 v_2 \dots v_n \neq \epsilon$, then we claim that

$$(6) \quad r(C(u, v), a) = \begin{cases} C(a, \epsilon) \cap C(u, v_2 v_3 \dots v_n) & \text{if } a = v_1 \\ \emptyset & \text{otherwise.} \end{cases}$$

To see this observe that every vertex in $C(u, v)$ emits exactly one edge, labelled v_1 , and so $r(C(u, v), a) = \emptyset$ if $v_1 \neq a$. If $v_1 = a$ and $x \in C(a, \epsilon) \cap C(u, v_2 v_3 \dots v_n)$, then $ax \in C(u, v)$ and so $x \in r(C(u, v), a)$. If $v_1 = a$ and $x \in r(C(u, v), a)$, then $ax \in C(u, v)$ and so $x \in C(a, \epsilon) \cap C(u, v_2 v_3 \dots v_n)$. Thus Equation (6) holds. If $u \in \mathcal{L}_X^\#(E_X)$ and $a \in \mathcal{A}$, then it is easy to check that $r(C(u, \epsilon), a) = C(ua, \epsilon)$. It follows that \mathcal{B}_X is accommodating for (E_X, \mathcal{L}_X) .

Since a vertex $x \in E_X^0$ can only receive an edge labelled a from the vertex ax we see that (E_X, \mathcal{L}_X) is left-resolving. It follows that $(E_X, \mathcal{L}_X, \mathcal{B}_X)$ is weakly left-resolving and regular. \square

As in [7], we let $\tilde{\mathcal{D}}_X$ denote the C^* -algebra generated by $\{\chi_{C(u, v)} : u, v \in \mathcal{L}_X^*(E_X)\}$. We then have that $\tilde{\mathcal{D}}_X = \overline{\text{span}}\{\chi_A : A \in \mathcal{B}_X\} = \mathcal{A}(E_X, \mathcal{L}_X, \mathcal{B}_X)$.

According to [7, Remark 7.3], $C^*(X)$ is the universal C^* -algebra generated by a family $\{S_a : a \in \mathcal{A}\}$ of partial isometries such that there exists a $*$ -homomorphism from $\iota_X : \tilde{\mathcal{D}}_X \rightarrow C^*(X)$ given by

$$(7) \quad \iota_X : \chi_{C(u, v)} \mapsto (S_{v_1} S_{v_2} \dots S_{v_m})(S_{u_1} S_{u_2} \dots S_{u_n})^* (S_{u_1} S_{u_2} \dots S_{u_n})(S_{v_1} S_{v_2} \dots S_{v_m})^*$$

where $u = u_1 u_2 \dots u_n$ and $v = v_1 v_2 \dots v_m \in \mathcal{L}_X^\#(E_X)$. Note that if u (respectively v) is the empty word, then the product $(S_{u_1} S_{u_2} \dots S_{u_n})$ (respectively $(S_{v_1} S_{v_2} \dots S_{v_m})$) is the identity of $C^*(X)$. If $u = u_1 u_2 \dots u_n \in \mathcal{L}_X^*(E_X)$, then we will denote the product $S_{u_1} S_{u_2} \dots S_{u_n}$ by S_u . If $u = \epsilon$, then we will denote the unit of $C^*(X)$ by S_u .

Lemma 7.5. *Let $\iota_X : \tilde{\mathcal{D}}_X \rightarrow C^*(X)$ be defined in (7) above. Then for every $x \in \tilde{\mathcal{D}}_X$ and $a \in \mathcal{A}$ we have $\iota_X(\phi_a(x)) = S_a^* \iota_X(x) S_a$ where $\phi_a : \tilde{\mathcal{D}}_X \rightarrow \tilde{\mathcal{D}}_X$ is given by $\phi_a(\chi_A) = \chi_{r(A, a)}$ for $A \in \mathcal{B}_X$ (see Lemma 5.1).*

Proof. Let $u, v \in \mathcal{L}_X^\#(E_X)$ with $v = v_1 v_2 \dots v_m \neq \epsilon$. If $a = v_1$ then

$$\begin{aligned} S_a^* \iota_X(\chi_{C(u, v)}) S_a &= S_a^* S_v S_u^* S_u S_v^* S_a = S_a^* S_a S_{v_2} S_{v_3} \dots S_{v_m} S_u^* S_u (S_{v_2} S_{v_3} \dots S_{v_m})^* \\ &= \iota_X(\chi_{C(a, \epsilon)} \chi_{C(u, v_2 v_3 \dots v_m)}), \end{aligned}$$

otherwise it is zero. Thus $S_a^* \iota_X(\chi_{C(u, v)}) S_a = \iota_X(\chi_{r(C(u, v), a)}) = \iota_X(\phi_a(\chi_{C(u, v)}))$ by Equation (6). One can similarly show that if $u \in \mathcal{L}_X^\#(E_X)$, then $S_a^* \iota_X(\chi_{C(u, \epsilon)}) S_a = \iota_X(\phi_a(\chi_{C(u, \epsilon)}))$. Since $\tilde{\mathcal{D}}_X$ is generated by $\{\chi_{C(u, v)} : u, v \in \mathcal{L}_X^\#(E_X)\}$, it follows that $\iota_X(\phi_a(x)) = S_a^* \iota_X(x) S_a$ for every $x \in \tilde{\mathcal{D}}_X$. \square

Proposition 7.6. *Let X be a one-sided shift space over a finite alphabet \mathcal{A} . Let $\{p_A, s_a : A \in \mathcal{B}_X, a \in \mathcal{A}\}$ and $\{S_a : a \in \mathcal{A}\}$ be the canonical generators of $C^*(E_X, \mathcal{L}_X, \mathcal{B}_X)$ and $C^*(X)$, respectively. Then the map $S_a \mapsto s_a$ for $a \in \mathcal{A}$ induces an isomorphism from $C^*(X)$ to $C^*(E_X, \mathcal{L}_X, \mathcal{B}_X)$.*

Proof. Let ι_X denote (the unique) $*$ -homomorphism from $\tilde{\mathcal{D}}_X$ to $C^*(X)$ mapping $\chi_{C(u, v)}$ to

$$(S_{v_1} S_{v_2} \dots S_{v_m})(S_{u_1} S_{u_2} \dots S_{u_n})^* (S_{u_1} S_{u_2} \dots S_{u_n})(S_{v_1} S_{v_2} \dots S_{v_m})^*$$

for $u, v \in \mathcal{L}_X^\#(E_X)$. We will show that $\{\iota_X(\chi_A), S_a : A \in \mathcal{B}_X, a \in \mathcal{A}\}$ is a representation of $(E_X, \mathcal{L}_X, \mathcal{B}_X)$. First, one checks that $\{\iota_X(\chi_A) : A \in \mathcal{B}_X\}$ satisfies condition (i) of Definition 2.3. Second, if $a \in \mathcal{A}$ and $A \in \mathcal{B}_X$, then it follows from Lemma 7.5 that

$$\iota_X(\chi_A)S_a = \iota_X(\chi_A)S_aS_a^*S_a = S_aS_a^*\iota_X(\chi_A)S_a = S_a\iota_X(\phi_a(\chi_A)) = S_a\iota_X(\chi_{r(A,a)})$$

which shows that $\{\iota_X(\chi_A), S_a : A \in \mathcal{B}_X, a \in \mathcal{A}\}$ satisfies condition (ii) of Definition 2.3. If $a \in \mathcal{A}$, then $S_a^*S_a = \iota_X(\chi_{C(a,\epsilon)}) = \iota_X(\chi_{r(a)})$. If $a, b \in \mathcal{A}$ and $a \neq b$, then $S_a^*S_b = S_a^*\iota_X(\chi_{C(\epsilon,a)}\chi_{C(\epsilon,b)})S_b = 0$. Thus $\{\iota_X(\chi_A), S_a : A \in \mathcal{B}_X, a \in \mathcal{A}\}$ satisfies condition (iii) of Definition 2.3. We have that $\bigcup_{a \in \mathcal{A}} C(\epsilon, a) = E_X^0$ and $C(\epsilon, a) \cap C(\epsilon, b) = \emptyset$ for $a, b \in \mathcal{A}$ with $a \neq b$ and so $\chi_{E_X^0} = \sum_{a \in \mathcal{L}_X(E_X^1)} \chi_{C(\epsilon,a)}$ in $\tilde{\mathcal{D}}_X$. Since $\iota_X(\chi_{E_X^0}) = 1_{C^*(X)}$ it follows from Lemma 7.5 that if $A \in \mathcal{B}_X$, then

$$\begin{aligned} \iota_X(\chi_A) &= \sum_{a \in \mathcal{L}_X(E_X^1)} \iota_X(\chi_{C(\epsilon,a)}\chi_A\chi_{C(\epsilon,a)}) = \sum_{a \in \mathcal{L}_X(AE_X^1)} S_aS_a^*\iota_X(\chi_A)S_aS_a^* \\ &= \sum_{a \in \mathcal{L}_X(AE_X^1)} S_a\iota_X(\phi_a(\chi_A))S_a^* = \sum_{a \in \mathcal{L}_X(AE_X^1)} S_a\iota_X(\chi_{r(A,a)})S_a^* \end{aligned}$$

which shows that $\{\iota_X(\chi_A), S_a : A \in \mathcal{B}_X, a \in \mathcal{A}\}$ satisfies condition (iv) of Definition 2.3. Hence $\{\iota_X(\chi_A), S_a : A \in \mathcal{B}_X, a \in \mathcal{A}\}$ is a representation of $(E_X, \mathcal{L}_X, \mathcal{B}_X)$. It follows from the universal property of $C^*(E_X, \mathcal{L}_X, \mathcal{B}_X)$ that there is a *-homomorphism $\pi_{s,p} : C^*(E_X, \mathcal{L}_X, \mathcal{B}_X) \rightarrow C^*(X)$ which sends p_A to χ_A and s_a to S_a for all $A \in \mathcal{B}_X$ and $a \in \mathcal{A}$.

It follows from Lemma 3.1 that there exists a *-homomorphism from $\tilde{\mathcal{D}}_X = \mathcal{A}(E_X, \mathcal{L}_X, \mathcal{B}_X)$ to $C^*(E_X, \mathcal{L}_X, \mathcal{B}_X)$ which maps χ_A to p_A for every $A \in \mathcal{B}_X$. In particular, if $u, v \in \mathcal{L}_X^\#(E_X)$, then $\phi(\chi_{C(u,v)}) = p_{C(u,v)}$. Repeated applications of conditions (iii) and (iv) of Definition 2.3 show that if $u \in \mathcal{L}_X^\#(E_X)$ then $s_u^*s_u = p_{r(u)} = p_{C(u,\epsilon)}$ (if $u = \epsilon$, then $r(u) = C(u, \epsilon) = X$). For $v = v_1v_2 \dots v_m \in \mathcal{L}_X^*(E_X)$, condition (iv) of Definition 2.3 implies that

$$\begin{aligned} p_{C(u,v)} &= \sum_{a \in \mathcal{L}_X(C(u,v)E_X^1)} s_ap_{r(C(u,v),a)}s_a^* = s_{v_1}p_{C(u,v_2v_3 \dots v_m)}s_{v_1}^* \text{ by Equation (6)} \\ &= s_{v_1} \left(\sum_{a \in \mathcal{L}_X(C(u,v_2v_3 \dots v_m)E_X^1)} s_ap_{r(C(u,v_2v_3 \dots v_m),a)}s_a^* \right) s_{v_1}^* = s_{v_1}s_{v_2}p_{C(u,v_3 \dots v_m)}s_{v_2}^*s_{v_1}^* \\ &= \dots = s_{v_1}s_{v_2} \dots s_{v_m}p_{C(u,\epsilon)}s_{v_m}^* \dots s_{v_2}^*s_{v_1}^* = s_vs_u^*s_us_v^*. \end{aligned}$$

Thus, it follows from the universal property of $C^*(X)$ that there is a *-homomorphism $\pi_{s,p} : C^*(X) \rightarrow C^*(E_X, \mathcal{L}_X, \mathcal{B}_X)$ which sends S_a to s_a and $\iota_X(\chi_A)$ to p_A for all $a \in \mathcal{A}$ and $A \in \mathcal{B}_X$.

Since $\pi_{s,p}$ and $\pi_{s,p}$ are evidently inverses, the result follows. \square

Corollary 7.7 (cf. [10, Theorem 1.1]). *Let X be a one-sided shift space over a finite alphabet \mathcal{A} . Let $(1 - \Phi) : \text{span}_{\mathbb{Z}}\{\chi_A : A \in \mathcal{B}_X\} \rightarrow \text{span}_{\mathbb{Z}}\{\chi_A : A \in \mathcal{B}_X\}$ be the linear map given by*

$$(1 - \Phi)(\chi_A) = \chi_A - \sum_{a \in \mathcal{L}_X(AE_X^1)} \chi_{r(A,a)} \quad A \in \mathcal{B}_X.$$

Then $K_1(C^(X))$ is isomorphic to $\ker(1 - \Phi)$ and there exists an isomorphism from $K_0(C^*(X))$ to $\text{coker}(1 - \Phi)$ which maps $[S_u^*S_u]_0$ to $\chi_{r(u)} + \text{Im}(1 - \Phi)$ for each $u \in \mathcal{L}_X^*(E_X)$.*

Proof. Notice that $(\mathcal{B}_X)_J = \mathcal{B}_X$. The result now follows from Theorem 6.4 and Proposition 7.6. \square

8. COMPUTING K -THEORY FOR $C^*(E, \mathcal{L}, \mathcal{E}^{0,-})$

In this section we give a practical method for computing the K -theory of labelled graph algebras of the form $C^*(E, \mathcal{L}, \mathcal{E}^{0,-})$ for left-resolving labelled graphs (E, \mathcal{L}) with no sources and sinks such that $(E, \mathcal{L}, \mathcal{E}^{0,-})$ is a set-finite and receiver set-finite labelled space. We use Theorem 6.4 to give a description of the K -theory of $C^*(E, \mathcal{L}, \mathcal{E}^{0,-})$ as an inductive limit in analogy with the computations of [35].

Standing Assumption: Throughout this section (E, \mathcal{L}) shall be a left-resolving labelled graph with no sources and sinks such that $(E, \mathcal{L}, \mathcal{E}^{0,-})$ is a set-finite and receiver set-finite labelled space.

Before we begin, let us recall some notation from [5]: For $v \in E^0$ and $\ell \geq 1$ let

$$\Lambda_\ell(v) = \{\alpha \in \mathcal{L}(E^k) : k \leq \ell, v \in r(\alpha)\}$$

denote the set of labelled paths of length at most ℓ which have a representative which terminates at v . Define the relation \sim_ℓ on E^0 by $v \sim_\ell w$ if and only if $\Lambda_\ell(v) = \Lambda_\ell(w)$; hence $v \sim_\ell w$ if v and w receive exactly the same labelled paths of length at most ℓ . Evidently \sim_ℓ is an equivalence relation and we use $[v]_\ell$ to denote the equivalence class of $v \in E^0$. We call the $[v]_\ell$ *generalised vertices* as they play the same role in labelled spaces as vertices in a directed graph.

Set $\Omega_\ell = E^0 / \sim_\ell = \{[v]_\ell : v \in E^0\}$ and let

$$\Omega = \bigcup_{\ell=1}^{\infty} \Omega_\ell.$$

It follows from [5, Proposition 2.4] that if $v \in E^0$ and $\ell \geq 1$, then

$$(8) \quad [v]_\ell = \bigcup_{i=1}^n [w_i]_{\ell+1} \text{ for some } w_1, \dots, w_n \in [v]_\ell.$$

We then have that

$$(9) \quad r([v]_\ell, a) = \bigcup_{j=1}^m r([w_j]_{\ell+1}, a) \text{ for all } a \in \mathcal{A}.$$

We now replace the labelled space $(E, \mathcal{L}, \mathcal{E}^{0,-})$ by one which is easier to work with. Let $\tilde{\Omega}$ be the smallest subset of 2^{E^0} containing Ω which is closed under finite unions.

Lemma 8.1. *The collection $\tilde{\Omega}$ is accommodating for (E, \mathcal{L}) and $(E, \mathcal{L}, \tilde{\Omega})$ is weakly left-resolving and regular.*

Proof. By [5, Proposition 2.4 (ii)] to prove that $\tilde{\Omega}$ is accommodating it suffices to show that $\tilde{\Omega}$ is closed under intersections and relative ranges. Since nontrivial intersection in Ω only occurs through containment, $\tilde{\Omega}$ is automatically closed under finite intersections.

Let $[v]_\ell \in \Omega_\ell$ and $a \in \mathcal{L}(E^1)$, and let $w_1 \in r([v]_\ell, a)$. Then $[w_1]_{\ell+1} \subseteq r([v]_\ell, a)$ because all vertices in $[w_1]_{\ell+1}$ must receive the same labelled paths as w_1 , in particular, they must be in $r([v]_\ell, a)$. If $r([v]_\ell, a) \neq [w_1]_{\ell+1}$ we can find $w_2 \in r([v]_\ell, a)$ such that $w_2 \notin [w_1]_{\ell+1}$. Similarly, we have $[w_2]_{\ell+1} \subseteq r([v]_\ell, a)$. Since $(E, \mathcal{L}, \mathcal{E}^{0,-})$ is receiver set-finite it follows from [5, Remark 3.5] that $\{\alpha : \alpha \in \mathcal{L}(E^{\ell+1}), r(\alpha) \in r([v]_\ell, a)\}$ is finite. Hence there are vertices w_1, \dots, w_m in $r([v]_\ell, a)$ such that

$$(10) \quad r([v]_\ell, a) = \bigcup_{j=1}^m [w_j]_{\ell+1}.$$

This shows that $\tilde{\Omega}$ is closed under relative ranges, and thus that it is accommodating for (E, \mathcal{L}) .

Since (E, \mathcal{L}) is left-resolving, it follows that $(E, \mathcal{L}, \tilde{\Omega})$ is weakly left-resolving and regular. \square

Proposition 8.2. *The two C*-algebras $C^*(E, \mathcal{L}, \mathcal{E}^{0,-})$ and $C^*(E, \mathcal{L}, \tilde{\Omega})$ are canonically isomorphic.*

Proof. By [5, Proposition 2.4] we can see that $\tilde{\Omega}$ is equal to $\widetilde{\mathcal{E}^{0,-}}$. The result then follows from Proposition 3.2. \square

We let $\mathbb{Z}(\Omega)$ denote the subgroup $\text{span}_{\mathbb{Z}}\{\chi_{[v]_{\ell}} : [v]_{\ell} \in \Omega\}$ of the group of functions from E^0 to \mathbb{Z} .

Corollary 8.3. *Let $(1 - \Phi) : \mathbb{Z}(\Omega) \rightarrow \mathbb{Z}(\Omega)$ be the linear map defined by*

$$(1 - \Phi)(\chi_{[v]_{\ell}}) = \chi_{[v]_{\ell}} - \sum_{a \in \mathcal{L}([v]_{\ell} E^1)} \chi_{r([v]_{\ell}, a)} \text{ for } [v]_{\ell} \in \Omega.$$

Then

$$\begin{aligned} K_1(C^*(E, \mathcal{L}, \mathcal{E}^{0,-})) &= K_1(C^*(E, \mathcal{L}, \tilde{\Omega})) \cong \ker(1 - \Phi) \text{ and} \\ K_0(C^*(E, \mathcal{L}, \mathcal{E}^{0,-})) &= K_0(C^*(E, \mathcal{L}, \tilde{\Omega})) \cong \text{coker}(1 - \Phi) \end{aligned}$$

via $[p]_{[v]_{\ell}}]_0 \mapsto \chi_{[v]_{\ell}} + \text{Im}(1 - \Phi)$ for $[v]_{\ell} \in \Omega$.

Proof. Since $\hat{\tilde{\Omega}}_J = \hat{\tilde{\Omega}} = \tilde{\Omega}$ and $\text{span}_{\mathbb{Z}}\{\chi_{[v]_{\ell}} : [v]_{\ell} \in \tilde{\Omega}\} = \mathbb{Z}(\Omega)$, the result follows from Lemma 8.1, Proposition 8.2 and Theorem 6.4. \square

In Theorem 8.7 we shall show how to compute the kernel and cokernel of $(1 - \Phi)$ in terms of a certain inductive limit. For $\ell \geq 1$ define $\mathbb{Z}(\Omega_{\ell}) = \text{span}_{\mathbb{Z}}\{\chi_{[v]_{\ell}} : [v]_{\ell} \in \Omega_{\ell}\}$. Note that since for every $v \in E^0$ and $\ell \geq 1$ there are vertices $w_1, \dots, w_m \in [v]_{\ell}$ such that $[v]_{\ell} = \bigcup_{j=1}^m [w_j]_{\ell+1}$ we have a linear map $i_{\ell} : \mathbb{Z}(\Omega_{\ell}) \rightarrow \mathbb{Z}(\Omega_{\ell+1})$ defined by

$$i_{\ell}(\chi_{[v]_{\ell}}) = \sum_{j=1}^m \chi_{[w_j]_{\ell+1}}.$$

Therefore we have an inductive system $\varinjlim (\mathbb{Z}(\Omega_{\ell}), i_{\ell})$. Note that since, for fixed $\ell \geq 1$, the sets $[v]_{\ell}$ are disjoint, the functions $\chi_{[v]_{\ell}}$ form a basis for $\mathbb{Z}(\Omega_{\ell})$.

Proposition 8.4. *There is an isomorphism*

$$\Psi_{\infty} : \varinjlim (\mathbb{Z}(\Omega_{\ell}), i_{\ell}) \rightarrow \mathbb{Z}(\Omega)$$

such that $\Psi_{\infty}(\chi_{[v]_{\ell}}) = \chi_{[v]_{\ell}}$ for all $[v]_{\ell} \in \Omega$.

Proof. For each $\ell \geq 1$ define $\Psi_{\ell} : \mathbb{Z}(\Omega_{\ell}) \rightarrow \mathbb{Z}(\Omega)$ by $\Psi_{\ell}(\chi_{[v]_{\ell}}) = \chi_{[v]_{\ell}}$. One checks that $\Psi_{\ell} \circ (i_{\ell} \circ \dots \circ i_k) = \Psi_k$ for every $\ell < k$ and so there is a map $\Psi_{\infty} : \varinjlim (\mathbb{Z}(\Omega_{\ell}), i_{\ell}) \rightarrow \mathbb{Z}(\Omega)$. Since $\mathbb{Z}(\Omega) = \bigcup_{\ell \geq 1} \Psi_{\ell}(\mathbb{Z}(\Omega_{\ell}))$ and each Ψ_{ℓ} is injective, it follows that Ψ_{∞} is an isomorphism and our result is established. \square

It follows from Equation (10) that we can make the following definitions.

Definition 8.5. For $\ell \geq 1$ define a linear map $(1 - \Phi)_{\ell} : \mathbb{Z}(\Omega_{\ell}) \rightarrow \mathbb{Z}(\Omega_{\ell+1})$ by

$$(1 - \Phi)_{\ell}(\chi_{[v]_{\ell}}) = i_{\ell}(\chi_{[v]_{\ell}}) - \sum_{a \in \mathcal{L}([v]_{\ell} E^1)} \chi_{r([v]_{\ell}, a)},$$

and define a linear map $(1 - \Phi) : \mathbb{Z}(\Omega) \rightarrow \mathbb{Z}(\Omega)$ by

$$(1 - \Phi)(\chi_{[v]_\ell}) = \chi_{[v]_\ell} - \sum_{a \in \mathcal{L}([v]_\ell E^1)} \chi_{r([v]_\ell, a)}.$$

Lemma 8.6. *For all $\ell \geq 1$, we have $(1 - \Phi)_{\ell+1} \circ i_\ell = i_{\ell+1} \circ (1 - \Phi)_\ell$. Therefore the $(1 - \Phi)_\ell$'s induce a map $(1 - \Phi)_\infty : \varinjlim (\mathbb{Z}(\Omega_\ell), i_\ell) \rightarrow \varinjlim (\mathbb{Z}(\Omega_\ell), i_\ell)$ which satisfies $(1 - \Phi) \circ \Psi_\infty = \Psi_\infty \circ (1 - \Phi)_\infty$.*

Proof. Let $\ell \geq 1$. Then for $[v]_\ell \in \Omega_\ell$ with $[v]_\ell = \bigcup_{j=1}^m [w_j]_{\ell+1}$ we have

$$\begin{aligned} (1 - \Phi)_{\ell+1}(i_\ell(\chi_{[v]_\ell})) &= (1 - \Phi)_{\ell+1}\left(\sum_{j=1}^m \chi_{[w_j]_{\ell+1}}\right) \\ &= \sum_{j=1}^m \left(i_{\ell+1}(\chi_{[w_j]_{\ell+1}}) - \sum_{a \in \mathcal{L}([w_j]_{\ell+1} E^1)} \chi_{r([w_j]_{\ell+1}, a)} \right). \end{aligned}$$

Also,

$$\begin{aligned} i_{\ell+1}((1 - \Phi)_\ell(\chi_{[v]_\ell})) &= i_{\ell+1}\left(i_\ell(\chi_{[v]_\ell}) - \sum_{a \in \mathcal{L}([v]_\ell E^1)} \chi_{r([v]_\ell, a)}\right) \\ &= i_{\ell+1}\left(\sum_{j=1}^m \chi_{[w_j]_{\ell+1}} - \sum_{j=1}^m \sum_{a \in \mathcal{L}([w_j]_{\ell+1} E^1)} \chi_{r([w_j]_{\ell+1}, a)}\right) \end{aligned}$$

since $\chi_{r([v]_\ell, a)} = \sum_{j=1}^m \chi_{r([w_j]_{\ell+1}, a)}$,

$$= \sum_{j=1}^m \left(i_{\ell+1}(\chi_{[w_j]_{\ell+1}}) - \sum_{a \in \mathcal{L}([w_j]_{\ell+1} E^1)} \chi_{r([w_j]_{\ell+1}, a)} \right)$$

as $i_{\ell+1}$ is really just the identity map on $\Omega_{\ell+1}$ and so $(1 - \Phi)_{\ell+1} \circ i_\ell = i_{\ell+1} \circ (1 - \Phi)_\ell$.

Also for $\ell \geq 1$ we have

$$(1 - \Phi)(\Psi_\infty(\chi_{[v]_\ell})) = (1 - \Phi)(\chi_{[v]_\ell}) = i_\ell(\chi_{[v]_\ell}) - \sum_{a \in \mathcal{L}([v]_\ell E^1)} \chi_{r([v]_\ell, a)}$$

and

$$\begin{aligned} \Psi_\infty((1 - \Phi)_\infty(\chi_{[v]_\ell})) &= \Psi_\infty((1 - \Phi)_\ell(\chi_{[v]_\ell})) = \Psi_\infty\left(i_\ell(\chi_{[v]_\ell}) - \sum_{a \in \mathcal{L}([v]_\ell E^1)} \chi_{r([v]_\ell, a)}\right) \\ &= i_\ell(\chi_{[v]_\ell}) - \sum_{a \in \mathcal{L}([v]_\ell E^1)} \chi_{r([v]_\ell, a)} \end{aligned}$$

and so $(1 - \Phi) \circ \Psi_\infty = \Psi_\infty \circ (1 - \Phi)_\infty$. □

We have established the following commuting diagram:

$$\begin{array}{ccccccc}
 \mathbb{Z}(\Omega_1) & \xrightarrow{i_1} & \mathbb{Z}(\Omega_2) & \xrightarrow{i_2} & \mathbb{Z}(\Omega_2) & \xrightarrow{i_3} & \dots & \xrightarrow{\lim} & \mathbb{Z}(\Omega_\ell, i_\ell) & \xrightarrow{\Psi_\infty} & \mathbb{Z}(\Omega) \\
 & & \searrow (1-\Phi)_1 & & \searrow (1-\Phi)_2 & & & & \downarrow (1-\Phi)_\infty & & \downarrow 1-\Phi \\
 \mathbb{Z}(\Omega_1) & \xrightarrow{i_1} & \mathbb{Z}(\Omega_2) & \xrightarrow{i_2} & \mathbb{Z}(\Omega_2) & \xrightarrow{i_3} & \dots & \xrightarrow{\lim} & \mathbb{Z}(\Omega_\ell, i_\ell) & \xrightarrow{\Psi_\infty} & \mathbb{Z}(\Omega)
 \end{array}$$

Let $\ell \geq 1$. It is easy to check that $i_\ell(\ker(1-\Phi)_\ell) \subseteq \ker(1-\Phi)_{\ell+1}$ and $i_\ell(\text{Im}(1-\Phi)_\ell) \subseteq \text{Im}(1-\Phi)_{\ell+1}$. It follows that $i_\ell : \mathbb{Z}(\Omega_\ell) \rightarrow \mathbb{Z}(\Omega_{\ell+1})$ induces maps $(i_\ell)_* : \ker(1-\Phi)_\ell \rightarrow \ker(1-\Phi)_{\ell+1}$ and $\tilde{i}_{\ell+1} : \text{coker}(1-\Phi)_\ell \rightarrow \text{coker}(1-\Phi)_{\ell+1}$.

Theorem 8.7. *With the above notation we have*

$$(11) \quad \ker(1-\Phi) \cong \varinjlim (\ker(1-\Phi)_\ell, (i_\ell)_*),$$

$$(12) \quad \text{coker}(1-\Phi) \cong \varinjlim (\text{coker}(1-\Phi)_\ell, \tilde{i}_\ell).$$

Proof. By definition of $(1-\Phi)_\infty$ we have

$$\ker(1-\Phi)_\infty \cong \varinjlim (\ker(1-\Phi)_\ell, (i_\ell)_*).$$

It follows from Lemma 8.6 that $\ker(1-\Phi) \cong \ker(1-\Phi)_\infty$ and Equation (11) follows. We claim that

$$\text{coker}(1-\Phi)_\infty \cong \varinjlim (\text{coker}(1-\Phi)_\ell, \tilde{i}_\ell).$$

Define $\eta_\ell : \text{coker}(1-\Phi)_\ell \rightarrow \text{coker}(1-\Phi)_\infty$ by

$$\eta_\ell (\chi_{[v]_{\ell+1}} + \text{Im}(1-\Phi)_\ell) = \chi_{[v]_\ell} + \text{Im}(1-\Phi)_\infty.$$

Note that η_ℓ is well-defined since $\text{Im}(1-\Phi)_\ell \subseteq \text{Im}(1-\Phi)_\infty$ for all $\ell \geq 1$. We claim that $\eta_{\ell+1} \circ \tilde{i}_\ell = \eta_\ell$ for all $\ell \geq 1$. Since

$$\begin{aligned}
 \eta_{\ell+1} \tilde{i}_\ell (\chi_{[v]_{\ell+1}} + \text{Im}(1-\Phi)_\ell) &= \eta_{\ell+1} (i_{\ell+1} \chi_{[v]_{\ell+1}} + \text{Im}(1-\Phi)_{\ell+1}) \\
 &= i_{\ell+1} (\chi_{[v]_{\ell+1}}) + \text{Im}(1-\Phi)_\infty \\
 &= \chi_{[v]_{\ell+1}} + \text{Im}(1-\Phi)_\infty \\
 &= \eta_\ell (\chi_{[v]_{\ell+1}} + \text{Im}(1-\Phi)_\ell),
 \end{aligned}$$

this establishes our claim. By the universal property of the inductive limit the η_ℓ 's induce a map $\eta_\infty : \varinjlim (\text{coker}(1-\Phi)_\ell, \tilde{i}_\ell) \rightarrow \text{coker}(1-\Phi)_\infty$ which is injective since each η_ℓ is injective, and is surjective as $\text{coker}(1-\Phi)_\infty = \bigcup_{\ell \geq 1} \eta_\ell(\text{coker}(1-\Phi)_\ell)$. Hence η_∞ is an isomorphism which establishes our claim. Finally, it follows from Lemma 8.6 that $\text{coker}(1-\Phi)_\infty \cong \text{coker}(1-\Phi)$ and Equation (12) follows. \square

Remark 8.8. Recall from [5] that to a left-resolving labelled graph (E, \mathcal{L}) with no sources and sinks over a finite alphabet we may associate an essential symbolic matrix system $(M(E), I(E))$, which by [30, Proposition 2.1] determines a unique λ -graph system $\mathfrak{L}_{M(E), I(E)}$.

By [5, Proposition 3.6] we know that $C^*(E, \mathcal{L}, \mathcal{E}^{0,-})$ is isomorphic to $\mathcal{O}_{\mathfrak{L}_{M(E), I(E)}}$. We should therefore expect some similarities between our computation of the K -theory of $C^*(E, \mathcal{L}, \mathcal{E}^{0,-})$ and the computation of the K -theory of $\mathcal{O}_{\mathfrak{L}_{M(E), I(E)}}$ outlined in [30, Section 9] (see also [27, 33]). Indeed, this is the case.

To an essential symbolic matrix system $(M, I) = (M_{\ell, \ell+1}, I_{\ell, \ell+1})_{\ell \geq 1}$ we may associate a λ -graph system $\mathfrak{L}_{M, I}$ as in [30, Section 2]. Following [30, Section 9] we see that

$$K_*(\mathcal{O}_{\mathfrak{L}_{M, I}}) = K_*(M, I) = \varinjlim (K_*^\ell(M, I), i_*^\ell)$$

for $*$ = 0, 1 where

$$\begin{aligned} K_0^\ell(M, I) &= \mathbb{Z}^{m(\ell+1)} / (I_{\ell, \ell+1}^t - M_{\ell, \ell+1}^t) \mathbb{Z}^{m(\ell)} \\ K_1^\ell(M, I) &= \ker(I_{\ell, \ell+1}^t - M_{\ell, \ell+1}^t) \text{ in } \mathbb{Z}^{m(\ell)} \end{aligned}$$

and $i_*^\ell : K_*^\ell(M, I) \rightarrow K_*^{\ell+1}(M, I)$ is induced by the map $I_{\ell, \ell+1}^t : \mathbb{Z}^{m(\ell)} \rightarrow \mathbb{Z}^{m(\ell+1)}$.

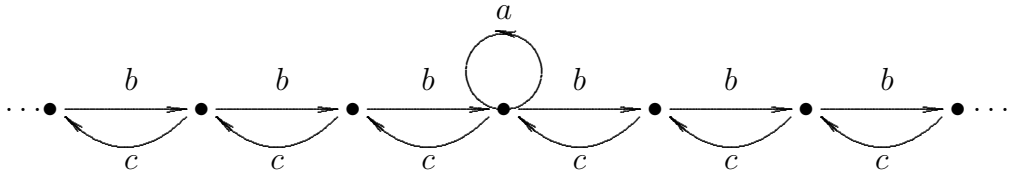
Under our identification of the labelled graph (E, \mathcal{L}) with the essential symbolic matrix system $(M(E), I(E))$ we have $|\Omega_\ell| = m(\ell)$, and so we may identify the group $\mathbb{Z}(\Omega_\ell)$ with $\mathbb{Z}^{m(\ell)}$ and the map $(1 - \Phi)_\ell : \mathbb{Z}(\Omega_\ell) \rightarrow \mathbb{Z}(\Omega_{\ell+1})$ with the map $(I(E)_{\ell, \ell+1}^t - M(E)_{\ell, \ell+1}^t) : \mathbb{Z}^{m(\ell)} \rightarrow \mathbb{Z}^{m(\ell+1)}$. Hence $\text{coker}(1 - \Phi)_\ell$ may be identified with $K_0^\ell(M(E), I(E))$ and $\ker(1 - \Phi)_\ell$ may be identified with $K_1^\ell(M(E), I(E))$. Since the map $(i_\ell)_* : \ker(1 - \Phi)_\ell \rightarrow \ker(1 - \Phi)_{\ell+1}$ corresponds to $i_0^\ell : K_0^\ell(M(E), I(E)) \rightarrow K_0^{\ell+1}(M(E), I(E))$ and the map $\tilde{i}_\ell : \text{coker}(1 - \Phi)_\ell \rightarrow \text{coker}(1 - \Phi)_{\ell+1}$ corresponds to $i_1^\ell : K_1^\ell(M(E), I(E)) \rightarrow K_1^{\ell+1}(M(E), I(E))$, we may identify $\text{coker}(1 - \Phi)$ with $K_0(M(E), I(E))$ and $\ker(1 - \Phi)$ with $K_1(M(E), I(E))$.

9. COMPUTATIONS

In this section we compute the K -theory of certain labelled graphs using the techniques outlined in Section 8.

Example 9.1. Let (E_F, \mathcal{L}_F) be the left Fischer cover (see [2, Example 2.15]) of the even shift. Then $\Omega_\ell = E_F^0$ for all $\ell \geq 1$ and so $\mathbb{Z}(\Omega) = \mathbb{Z}^2$ and the matrix of $(1 - \Phi)$ is $\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$. Hence the K -theory of the labelled graph is the same as the K -theory of the underlying graph, that is $K_0 = K_1 = \{0\}$. This should not be a surprise in the light of [4, Theorem 6.6]. We obtain similar conclusions for the left Krieger cover (E_K, \mathcal{L}_K) of the even shift, though in this case $\Omega_\ell = E_K^0$ for $\ell \geq 2$.

Example 9.2. Now consider the labelled graph (E, \mathcal{L}) shown below



which was discussed in [5, Section 7.2] and [18]. We label the vertices of E by the integers, where $0 = r(a)$.

Fix $\ell \geq 1$. We set $\text{rest}_\ell = \{\pm\ell, \pm(\ell+1), \dots\}$. Then $[n]_\ell = \{n\}$ for $|n| \leq \ell - 1$ and $[n]_\ell = \text{rest}_\ell$ for $|n| \geq \ell$. Hence, if we identify each vertex n with the singleton $\{n\}$, then we have that

$$\Omega_1 = \{0, \text{rest}_1\}, \Omega_2 = \{0, \pm 1, \text{rest}_2\}, \dots, \Omega_\ell = \{0, \pm 1, \pm 2, \dots, \pm(\ell - 1), \text{rest}_\ell\}, \dots$$

By Definition 8.5 we have $(1 - \Phi)_1(\chi_0) = -\chi_1 - \chi_{-1}$ and $(1 - \Phi)_1(\chi_{\text{rest}_1}) = -2\chi_0 - \chi_{\text{rest}_2}$. It is straightforward to show that $\ker(1 - \Phi)_1 = \{0\}$ and that every element of $\mathbb{Z}(\Omega_2)$ can be written as $a\chi_0 + b\chi_1 + w$ where $a, b, \in \mathbb{Z}$ and $w \in \text{Im}(1 - \Phi)_1$. Hence $\text{coker}(1 - \Phi)_1 \cong \mathbb{Z}^2$. For $\ell \geq 2$ the maps $(1 - \Phi)_\ell : \mathbb{Z}(\Omega_\ell) \rightarrow \mathbb{Z}(\Omega_{\ell+1})$ are given by

$$(13) \quad (1 - \Phi)_\ell(\chi_n) = \chi_n - \chi_{n-1} - \chi_{n+1} \text{ if } n \neq \text{rest}_\ell, 0$$

$$(14) \quad (1 - \Phi)_\ell(\chi_0) = -\chi_1 - \chi_{-1}$$

$$(15) \quad (1 - \Phi)_\ell(\chi_{\text{rest}_\ell}) = -\chi_{\ell-1} - \chi_{-\ell+1} - \chi_{\text{rest}_{\ell+1}}.$$

Through systematic use of (15), then (13) for $n = -\ell + 1, \dots, -1$, followed by (14) and then (13) for $n = \ell - 1, \dots, 1$, one checks that $\{(1 - \Phi)_\ell(\chi_i) : i \in \Omega_\ell\}$ is linearly independent in $\mathbb{Z}(\Omega_{\ell+1})$, and so $\ker(1 - \Phi)_\ell = \{0\}$ for $\ell \geq 2$. A similar procedure allows one to show that every element of $\mathbb{Z}(\Omega_\ell)$ can be written in the form $a\chi_0 + b\chi_1 + w$ where $a, b, \in \mathbb{Z}$ and $w \in \text{Im}(1 - \Phi)_\ell$. Hence for $\ell \geq 2$ $\text{coker}(1 - \Phi)_\ell = \mathbb{Z}(\Omega_{\ell+1}) / \text{Im}(1 - \Phi)_\ell$ is freely generated by

$$\chi_0 + \text{Im}(1 - \Phi)_\ell \text{ and } \chi_1 + \text{Im}(1 - \Phi)_\ell.$$

It follows that $\text{coker}(1 - \Phi)_\ell \cong \mathbb{Z}^2$ for all $\ell \geq 1$.

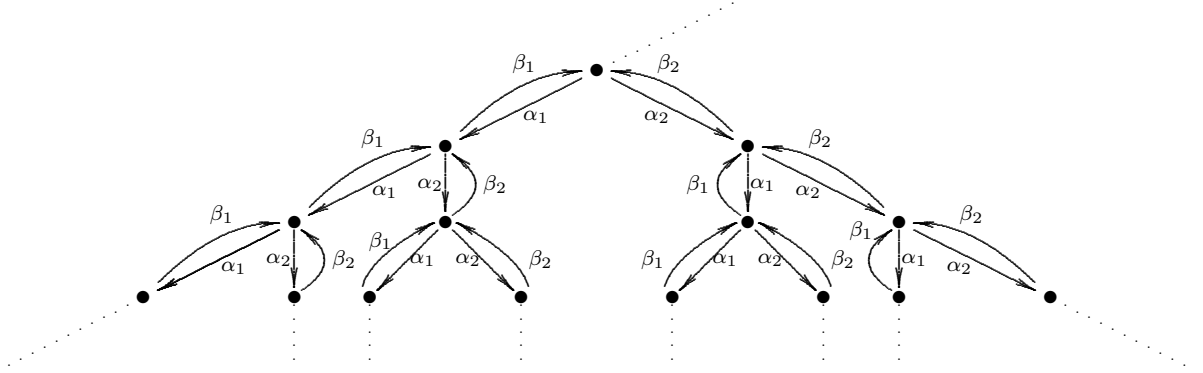
For $\ell \geq 1$ the maps $i_\ell : \mathbb{Z}(\Omega_\ell) \rightarrow \mathbb{Z}(\Omega_{\ell+1})$ are given by

$$\begin{aligned} i_\ell(\chi_i) &= \chi_i \text{ for } 0 \leq |i| \leq \ell - 1 \text{ and} \\ i_\ell(\chi_{\text{rest}_\ell}) &= \chi_\ell + \chi_{-\ell} + \chi_{\text{rest}_{\ell+1}} \end{aligned}$$

and so for $\ell \geq 2$ the maps \tilde{i}_ℓ are the identity map. Thus, it follows from Theorem 8.7 that

$$K_0(C^*(E, \mathcal{L}, \mathcal{E}^{0,-})) \cong \mathbb{Z}^2 \text{ and } K_1(C^*(E, \mathcal{L}, \mathcal{E}^{0,-})) \cong \{0\}.$$

Example 9.3. Consider the Dyck shift D_2 which has labelled graph presentation (E_2, \mathcal{L}_2) below:



Since every vertex receives an edge labelled β_1, β_2 , the vertices in Ω_ℓ are distinguished by which labelled paths of length ℓ involving the symbols α_1, α_2 they receive.

For a word w in the symbols α_1, α_2 , by an abuse of notation we denote the set of vertices comprising $r(w)$ by w . We then have

$$\begin{aligned} \Omega_1 &= \{\alpha_1, \alpha_2\} \\ \Omega_2 &= \{\alpha_1\alpha_1, \alpha_1\alpha_2, \alpha_2\alpha_1, \alpha_2\alpha_2\} \\ \Omega_3 &= \{\alpha_1\alpha_1\alpha_1, \alpha_1\alpha_1\alpha_2, \alpha_1\alpha_2\alpha_1, \alpha_1\alpha_2\alpha_2, \alpha_2\alpha_1\alpha_1, \alpha_2\alpha_1\alpha_2, \alpha_2\alpha_2\alpha_1, \alpha_2\alpha_2\alpha_2\} \\ &\vdots \\ \Omega_\ell &= \{\alpha_1^\ell, \alpha_1^{\ell-1}\alpha_2, \alpha_1^{\ell-2}\alpha_2\alpha_1, \alpha_1^{\ell-2}\alpha_2\alpha_2, \dots, \alpha_2^\ell\}. \end{aligned}$$

For $\ell = 1$ we have

$$\begin{aligned} (1 - \Phi)_1(\chi_{\alpha_1}) &= -\chi_{\alpha_1\alpha_1} - 2\chi_{\alpha_1\alpha_2} - \chi_{\alpha_2\alpha_2} \\ (1 - \Phi)_1(\chi_{\alpha_2}) &= -\chi_{\alpha_1\alpha_1} - 2\chi_{\alpha_2\alpha_1} - \chi_{\alpha_2\alpha_2}. \end{aligned}$$

It is straightforward to see that $\ker(1 - \Phi)_1 = \{0\}$. Let

$$x_{11} = \chi_{\alpha_1\alpha_1} + \text{Im}(1 - \Phi)_1, x_{12} = \chi_{\alpha_1\alpha_2} + \text{Im}(1 - \Phi)_1, x_{21} = \chi_{\alpha_2\alpha_1} + \text{Im}(1 - \Phi)_1, x_{22} = \chi_{\alpha_2\alpha_2} + \text{Im}(1 - \Phi)_1.$$

Then $2(x_{11} + x_{12} + x_{21} + x_{22}) = 0$ since

$$(1 - \Phi)_1(\chi_{\alpha_1} + \chi_{\alpha_2}) = 2(\chi_{\alpha_1\alpha_1} + \chi_{\alpha_1\alpha_2} + \chi_{\alpha_2\alpha_1} + \chi_{\alpha_2\alpha_2}).$$

It follows that $\text{coker}(1 - \Phi)_1 \cong \mathbb{Z}^2 \times (\mathbb{Z}/2\mathbb{Z})$, generated by $x_{12}, x_{22}, x_{11} + x_{12} + x_{21} + x_{22}$.

As every vertex emits an edge labelled α_1 and an edge labelled α_2 , we have $r(w, \alpha_i) = w\alpha_i$ for $i = 1, 2$ and all words w in the symbols α_1, α_2 . Furthermore $r(w\alpha_i, \beta_j) = w$ if $i = j$ and is empty if $i \neq j$. Since $[w]_\ell = [\alpha_1 w]_{\ell+1} \cup [\alpha_2 w]_{\ell+1}$ for all words $w \in \Omega_\ell$, we have $i_\ell(\chi_w) = \chi_{\alpha_1 w} + \chi_{\alpha_2 w}$, and so

$$\begin{aligned} (1 - \Phi)_2(\chi_{\alpha_1 \alpha_1}) &= -\chi_{\alpha_1 \alpha_1 \alpha_1} - \chi_{\alpha_1 \alpha_1 \alpha_2} - \chi_{\alpha_1 \alpha_2 \alpha_1} - \chi_{\alpha_2 \alpha_2 \alpha_1} \\ (1 - \Phi)_2(\chi_{\alpha_1 \alpha_2}) &= -\chi_{\alpha_1 \alpha_1 \alpha_1} + \chi_{\alpha_1 \alpha_1 \alpha_2} - 2\chi_{\alpha_1 \alpha_2 \alpha_1} - \chi_{\alpha_1 \alpha_2 \alpha_2} - \chi_{\alpha_2 \alpha_1 \alpha_1} + \chi_{\alpha_2 \alpha_1 \alpha_2} - \chi_{\alpha_2 \alpha_2 \alpha_1} \\ (1 - \Phi)_2(\chi_{\alpha_2 \alpha_1}) &= -\chi_{\alpha_1 \alpha_1 \alpha_2} + \chi_{\alpha_1 \alpha_2 \alpha_1} - \chi_{\alpha_1 \alpha_2 \alpha_2} - \chi_{\alpha_2 \alpha_1 \alpha_1} - 2\chi_{\alpha_2 \alpha_1 \alpha_2} + \chi_{\alpha_2 \alpha_2 \alpha_1} - \chi_{\alpha_2 \alpha_2 \alpha_2} \\ (1 - \Phi)_2(\chi_{\alpha_2 \alpha_2}) &= -\chi_{\alpha_1 \alpha_1 \alpha_2} - \chi_{\alpha_2 \alpha_1 \alpha_2} - \chi_{\alpha_2 \alpha_2 \alpha_1} - \chi_{\alpha_2 \alpha_2 \alpha_2}. \end{aligned}$$

One checks that $\ker(1 - \Phi)_2 = \{0\}$. For a word $w \in \Omega_3$, let $x_w = \chi_w + \text{Im}(1 - \Phi)_2$. Then $2 \sum_{w \in \Omega_3} x_w = 0$, because

$$(1 - \Phi)_2(\chi_{\alpha_1 \alpha_2} + \chi_{\alpha_2 \alpha_2} + \chi_{\alpha_1 \alpha_1} + \chi_{\alpha_2 \alpha_2}) = 2 \sum_{w \in \Omega_3} \chi_w.$$

It follows that $\text{coker}(1 - \Phi)_2 \cong \mathbb{Z}^4 \times (\mathbb{Z}/2\mathbb{Z})$, generated by $x_{112}, x_{122}, x_{212}, x_{222}, \sum_{w \in \Omega_3} x_w$.

Since $\tilde{i}_2(x_{12}) = x_{112} + x_{212}$, $\tilde{i}_2(x_{22}) = x_{122} + x_{222}$ and $\tilde{i}_2(x_{11} + x_{12} + x_{21} + x_{22}) = \sum_{w \in \Omega_3} x_w$ we see that $\tilde{i}_2 : \mathbb{Z}^2 \times (\mathbb{Z}/2\mathbb{Z}) \rightarrow \mathbb{Z}^4 \times (\mathbb{Z}/2\mathbb{Z})$ is the identity on the cyclic group of order 2 and an injection on the free abelian part.

In general we have for a word w of length n in Ω_n that

$$(1 - \Phi)_n(\chi_w) = \chi_{\alpha_1 w} + \chi_{\alpha_2 w} - \chi_{w\alpha_1} - \chi_{w\alpha_2} - \chi_{\alpha_1 \alpha_1 w_{n-1}} - \chi_{\alpha_2 \alpha_1 w_{n-1}} - \chi_{\alpha_1 \alpha_2 w_{n-1}} - \chi_{\alpha_2 \alpha_2 w_{n-1}}$$

where w_{n-1} represents the first $n - 1$ symbols of w . Again, a short calculation shows that $\ker(1 - \Phi)_n = \{0\}$. For a word $w \in \Omega_{n+1}$, let $x_w = \chi_w + \text{Im}(1 - \Phi)_n$. Then $2 \sum_{w \in \Omega_{n+1}} x_w = 0$, because

$$(1 - \Phi)_n \left(\sum_{w' \in \Omega_n} \chi_{w'} \right) = 2 \sum_{w \in \Omega_{n+1}} \chi_w.$$

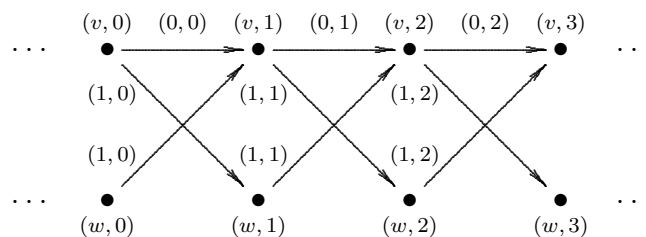
Using this, one can show that $\text{coker}(1 - \Phi)_n \cong \mathbb{Z}^{2^n} \times (\mathbb{Z}/2\mathbb{Z})$, generated by $\{x_{w'2} : w' \in \Omega_n\} \cup \{\sum_{w \in \Omega_{n+1}} x_w\}$.

Observe that \tilde{i}_n is induced by the map i_n which only adds symbols to the beginning of words. It follows that $\tilde{i}_n : \mathbb{Z}^{2^n} \times (\mathbb{Z}/2\mathbb{Z}) \rightarrow \mathbb{Z}^{2^{n+1}} \times (\mathbb{Z}/2\mathbb{Z})$ is the identity on the cyclic group of order 2 and an injection on the free abelian part. From Theorem 8.7 we have that

$$K_0(C^*(E_2, \mathcal{L}_2, \mathcal{E}_2^{0,-})) \cong \left(\prod_{i=1}^{\infty} \mathbb{Z} \right) \times (\mathbb{Z}/2\mathbb{Z}) \text{ and } K_1(C^*(E_2, \mathcal{L}_2, \mathcal{E}_2^{0,-})) \cong 0.$$

This confirms the calculations done in [23, §3]. It is worth noting that $C^*(E_2, \mathcal{L}_2, \mathcal{E}_2^{0,-})$ is unital (the unit is $p_{r(\alpha_1)} + p_{r(\alpha_2)}$) and has a K_0 -group which is not finitely generated, and so cannot be isomorphic to a graph algebra.

Example 9.4. Consider the following left-resolving labelled graph (E, \mathcal{L}) over the infinite alphabet $\{0, 1\} \times \mathbb{Z}$:



We write $E^0 = \{(v, i), (w, i) : i \in \mathbb{Z}\}$, and label the edges as shown. Since (v, i) is the only vertex which receives an edge labelled $(0, i - 1)$, we have $[(v, i)]_\ell = \{(v, i)\}$ for all $i \in \mathbb{Z}$ and $\ell \geq 1$. Since (w, i) is the only vertex which receives an edge with label $(1, i - 1)$, but does not receive any edge with label $(0, j)$ for any $j \in \mathbb{Z}$, we have $[(w, i)]_\ell = \{(w, i)\}$ for all $i \in \mathbb{Z}$ and $\ell \geq 1$. We will simply identify $[(v, i)]_\ell = \{(v, i)\}$ with (v, i) and $[(w, i)]_\ell = \{(w, i)\}$ with (w, i) for all $i \in \mathbb{Z}$. We then have that $\Omega_\ell = E^0$ for all $\ell \geq 1$.

Since $\Omega_\ell = E^0$ for each $\ell \geq 1$, and E^0 is infinite, we have

$$\mathbb{Z}(\Omega_\ell) = \bigoplus_{E^0} \mathbb{Z} = \text{span}_{\mathbb{Z}}\{\chi_{(v,i)}, \chi_{(w,i)} : i \in \mathbb{Z}\}.$$

Since (v, i) only connects to $(v, i + 1)$ and $(w, i + 1)$ and (w, i) only connects to $(v, i + 1)$ for all $i \in \mathbb{Z}$, we have

$$(16) \quad (1 - \Phi)_\ell(\chi_{(v,i)}) = \chi_{(v,i)} - \chi_{(v,i+1)} - \chi_{(w,i+1)} \text{ and}$$

$$(17) \quad (1 - \Phi)_\ell(\chi_{(w,i)}) = \chi_{(w,i)} - \chi_{(v,i+1)}.$$

We claim that $\ker(1 - \Phi)_\ell = \{0\}$ for all $\ell \geq 1$.

Suppose, for contradiction, that $b \in \ker((1 - \Phi)_\ell) \subseteq \bigoplus_{E^0} \mathbb{Z}$ is a nonzero vector. Since $b = \sum_{i \in \mathbb{Z}} b_{(v,i)} \chi_{(v,i)} + b_{(w,i)} \chi_{(w,i)}$ is in the direct sum there must be a maximum i such that not both $b_{(v,i)}$ and $b_{(w,i)}$ are 0. Then by (16) we have $b_{(v,i)} = b_{(v,i+1)} + b_{(w,i+1)} = 0$, and by (17) we have $b_{(w,i)} = b_{(v,i+1)} = 0$ which is a contradiction. Hence $\ker(1 - \Phi)_\ell = \{0\}$ for all $\ell \geq 1$.

Now we compute the cokernel of $(1 - \Phi)_\ell$. From Equation (16) and (17) we see that

$$(18) \quad (1 - \Phi)_\ell(\chi_{(v,i)} - \chi_{(w,i)}) = \chi_{(v,i)} - \chi_{(w,i)} - \chi_{(w,i+1)}.$$

Repeated use of Equation (18) shows that every vector in $\mathbb{Z}(\Omega_\ell)$ is equivalent, using vectors in $\text{Im}(1 - \Phi)_\ell$, to a vector which has all (v, i) -coordinates zero. From Equations (16) and (17) we also see that

$$(19) \quad (1 - \Phi)_\ell(\chi_{(v,i)} + \chi_{(w,i-1)} - \chi_{(w,i)}) = \chi_{(w,i-1)} - \chi_{(w,i)} - \chi_{(w,i+1)}.$$

Repeated use of Equation (19) shows that every vector in $\mathbb{Z}(\Omega_\ell)$ is equivalent, using vectors in $\text{Im}(1 - \Phi)_\ell$, to a vector which has all (v, i) -coordinates zero and all (w, i) -coordinates zero except for $(w, 0)$ and $(w, 1)$, with no relation between them. Hence $\text{coker}(1 - \Phi)_\ell$ is generated by $\chi_{(w,0)} + \text{Im}(1 - \Phi)_\ell$ and $\chi_{(w,1)} + \text{Im}(1 - \Phi)_\ell$ for all $\ell \geq 1$ and so we may conclude that $\text{coker}(1 - \Phi)_\ell \cong \mathbb{Z}^2$ for all $\ell \geq 1$.

Since \tilde{i}_ℓ is the identity map for all $\ell \geq 1$, we have

$$K_0(C^*(E, \mathcal{L}, \mathcal{E}^{0,-})) \cong \mathbb{Z}^2 \text{ and } K_1(C^*(E, \mathcal{L}, \mathcal{E}^{0,-})) \cong \{0\}.$$

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